

Paley-Wiener spaces with vanishing conditions and Painlevé VI transcendents

Jean-François Burnol

July 27, 2010

Abstract

We modify the classical Paley-Wiener spaces PW_x of entire functions of finite exponential type at most $x > 0$, which are square integrable on the real line, via the additional condition of vanishing at finitely many complex points z_1, \dots, z_n . We compute the reproducing kernels and relate their variations with respect to x to a Krein differential system, whose coefficient (which we call the μ -function) and solutions have determinantal expressions. Arguments specific to the case where the “trivial zeros” z_1, \dots, z_n are in arithmetic progression on the imaginary axis allow us to establish for expressions arising in the theory a system of two non-linear first order differential equations. A computation, having this non-linear system at his start, obtains quasi-algebraic and among them rational Painlevé transcendents of the sixth kind as certain quotients of such μ -functions.

Université Lille 1
UFR de Mathématiques
Cité scientifique M2
F-59655 Villeneuve d’Ascq
France
burnol@math.univ-lille1.fr

1 Introduction and summary of results

Let $\phi \in L^2(\mathbb{R}, dt)$ and $\mathcal{F}(\phi)(z) = \int_{-\infty}^{\infty} \phi(t)e^{izt} dt$ its Fourier transform. When ϕ is supported in $(-x, x)$, $f(z) = \mathcal{F}(\phi)(z)$ is an entire function of exponential type at most x . Conversely the Paley-Wiener theorem identifies the vector space PW_x of entire functions of exponential type at most x , square-integrable on the real line, as the Hilbert space of such Fourier transforms. Our convention for our scalar products is for them to be conjugate linear in the first factor and complex linear in the second factor. Specifically $(\phi, \psi) = \int_{\mathbb{R}} \overline{\phi(t)}\psi(t) dt$, hence for the transforms f and g : $(f, g) = \frac{1}{2\pi} \int_{\mathbb{R}} \overline{f(z)}g(z)dz = (\phi, \psi)$.

The evaluator Z_z is the element of PW_x such that

$$\forall g \in PW_x, g = \mathcal{F}(\psi), \quad (Z_z, g) = g(z) = \int_{-x}^x e^{izt} \psi(t) dt = (\mathcal{F}(e^{-i\bar{z}t}), g) \quad (1)$$

Hence:

$$Z_z(w) = 2 \frac{\sin((\bar{z} - w)x)}{\bar{z} - w} = \frac{e^{i\bar{z}x} e^{-iwx} - e^{-i\bar{z}x} e^{iwx}}{i(\bar{z} - w)} \quad (2)$$

Let $E(w) = e^{-iwx}$ and $E^*(w) = \overline{E(\bar{w})}$. The evaluators in PW_x are given by

$$Z_z(w) = (Z_w, Z_z) = \frac{\overline{E(z)}E(w) - \overline{E^*(z)}E^*(w)}{i(\bar{z} - w)} \quad (3)$$

Let us also define:

$$A(w) = \frac{1}{2}(E(w) + E^*(w)) \quad (4a)$$

$$B(w) = \frac{i}{2}(E(w) - E^*(w)) \quad (4b)$$

Then $E = A - iB$, $A = A^*$, $B = B^*$ and:

$$(Z_w, Z_z) = Z_z(w) = 2 \frac{\overline{B(z)}A(w) - \overline{A(z)}B(w)}{\bar{z} - w} \quad (5)$$

For the Paley-Wiener spaces, $A(w) = \cos(xw)$ is even and $B(w) = \sin(xw)$ is odd.

Let us consider generally a Hilbert space H , whose vectors are entire functions, and such that the evaluations at complex numbers are continuous linear forms, hence correspond to specific vectors Z_z . Let $\sigma = (z_1, \dots, z_n)$ be a finite sequence of distinct complex numbers. We let H^σ be the closed subspace of H of functions vanishing at the z_i 's. Let

$$\gamma(z) = \frac{1}{(z - z_1) \dots (z - z_n)} \quad (6)$$

and define $H(\sigma) = \gamma(z)H^\sigma$:

$$H(\sigma) = \{F(z) = \gamma(z)f(z) \mid f \in H, f(z_1) = \dots = f(z_n) = 0\} \quad (7)$$

We introduced this notion in [5]. We say that $F(z) = \gamma(z)f(z)$ is the “complete” form of f , and refer to z_1, \dots, z_n as the “trivial zeros” of f . We give $H(\sigma)$ the Hilbert space structure which makes $f \mapsto F$ an isometry with H^σ . Let us note that evaluations $F \mapsto F(z)$ are again continuous linear forms on this new Hilbert space of entire functions: this is immediate if $z \notin \sigma$ and follows from the Banach-Steinhaus theorem if $z \in \sigma$. We thus define \mathcal{K}_z in $H(\sigma)$ to be the evaluator at z :

$$\mathcal{K}_z \in H(\sigma) \quad \forall F \in H(\sigma) \quad F(z) = (\mathcal{K}_z, F)_{H(\sigma)} \quad (8)$$

Here is a summary of the results presented here. We start by showing how to find entire functions \mathcal{A}_σ and \mathcal{B}_σ , real on the real line, such that:

$$(\mathcal{K}_w, \mathcal{K}_z) = \mathcal{K}_z(w) = 2 \frac{\overline{\mathcal{B}_\sigma(z)} \mathcal{A}_\sigma(w) - \overline{\mathcal{A}_\sigma(z)} \mathcal{B}_\sigma(w)}{\bar{z} - w} \quad (9)$$

This will be done under the following hypotheses: (1) the initial Hilbert space of entire functions H satisfies the axioms of [2], hence its evaluators Z_z are given by a formula (5) for some entire functions A and B which are real on the real line, (2) A can be chosen even and B can be chosen odd, and (3) the added “trivial zeros” are purely imaginary. The produced functions \mathcal{A}_σ and \mathcal{B}_σ giving the reproducing kernel (9) of the modified space $H(\sigma)$ will be respectively even and odd. The restrictive hypotheses (2) and (3) can be disposed of, as is explained in companion paper [5]. We follow here another method, which proves formulas of a different type than those available from the general treatment [5]. The interested reader will find in [5] the easy arguments establishing that $H(\sigma)$ verifies the axioms of [2] if the initial space H does: this explains a priori why indeed a formula of the type (9) has to exist if (5) holds for H .

Then, we examine the case of a dependency of the initial space on a parameter x . Assuming that the initial A_x and B_x obey a first order differential system of the Krein type [10, 11] as functions of x (involving as coefficient what we call a μ -function) we prove that the new $\mathcal{A}_{\sigma,x}$ and $\mathcal{B}_{\sigma,x}$ do as well (in other words we compute the μ_σ -function in terms of the initial μ -function). The result is already notable when we start from the classical Paley-Wiener spaces for which the initial $\mu(x)$ -function ($x > 0$) vanishes identically. It will be achieved through establishing a “pre-Crum formula” for the effect of Darboux transformations on Schrödinger equations linked into Krein systems.

The final part of the paper establishes the main result. We consider the classical Paley-Wiener spaces PW_x modified by imaginary trivial zeros in an arithmetic progression σ . We prove that certain quotients of the μ -functions associated to the spaces $PW_x(\sigma)$ obey the Painlevé VI differential equation.

2 A determinantal identity

The following identity is quasi identical with a formula of Okada [12, Theorem 4.2] and immediately equivalent to it. We give a different proof.

Theorem 1. *Let there be given indeterminates $u_i, v_i, k_i, x_i, y_i, l_i$, for $1 \leq i \leq n$. We*

define the following $n \times n$ matrices

$$U_n = \begin{pmatrix} u_1 & u_2 & \dots & u_n \\ k_1 v_1 & k_2 v_2 & \dots & k_n v_n \\ k_1^2 u_1 & k_2^2 u_2 & \dots & k_n^2 u_n \\ \vdots & \dots & \dots & \vdots \end{pmatrix} \quad V_n = \begin{pmatrix} v_1 & v_2 & \dots & v_n \\ k_1 u_1 & k_2 u_2 & \dots & k_n u_n \\ k_1^2 v_1 & k_2^2 v_2 & \dots & k_n^2 v_n \\ \vdots & \dots & \dots & \vdots \end{pmatrix} \quad (10)$$

where the rows contain alternatively u 's and v 's. Similarly:

$$X_n = \begin{pmatrix} x_1 & x_2 & \dots & x_n \\ l_1 y_1 & l_2 y_2 & \dots & l_n y_n \\ l_1^2 x_1 & l_2^2 x_2 & \dots & l_n^2 x_n \\ \vdots & \dots & \dots & \vdots \end{pmatrix} \quad Y_n = \begin{pmatrix} y_1 & y_2 & \dots & y_n \\ l_1 x_1 & l_2 x_2 & \dots & l_n x_n \\ l_1^2 y_1 & l_2^2 y_2 & \dots & l_n^2 y_n \\ \vdots & \dots & \dots & \vdots \end{pmatrix} \quad (11)$$

There holds

$$\det_{1 \leq i, j \leq n} \left(\frac{u_i y_j - v_i x_j}{l_j - k_i} \right) = \frac{1}{\prod_{i,j} (l_j - k_i)} \begin{vmatrix} U_n & X_n \\ V_n & Y_n \end{vmatrix}_{2n \times 2n} \quad (12)$$

Proof. Let A, B, C, D be $n \times n$ matrices, with A and C invertible. Using $\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix} \begin{pmatrix} I & A^{-1}B \\ I & C^{-1}D \end{pmatrix}$ we obtain

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |A||C||C^{-1}D - A^{-1}B| \quad (13)$$

where vertical bars denote determinants. Let $d(u) = \text{diag}(u_1, \dots, u_n)$ and $p_u = \prod_{1 \leq i \leq n} u_i$. We define similarly $d(v), d(x), d(y)$ and p_v, p_x, p_y . From the previous identity we get

$$\begin{aligned} \begin{vmatrix} Ad(u) & Bd(x) \\ Cd(v) & Dd(y) \end{vmatrix} &= |A||C| p_u p_v \begin{vmatrix} d(v)^{-1} C^{-1} D d(y) - d(u)^{-1} A^{-1} B d(x) \end{vmatrix} \\ &= |A||C| \begin{vmatrix} d(u) C^{-1} D d(y) - d(v) A^{-1} B d(x) \end{vmatrix} \end{aligned} \quad (14)$$

The special case $A = C, B = D$, gives

$$\begin{vmatrix} Ad(u) & Bd(x) \\ Ad(v) & Bd(y) \end{vmatrix}_{2n \times 2n} = \det(A)^2 \det_{1 \leq i, j \leq n} ((u_i y_j - v_i x_j)(A^{-1}B)_{ij}) \quad (15)$$

Let $W(k)$ be the Vandermonde matrix with rows $(1 \dots 1), (k_1 \dots k_n), (k_1^2 \dots k_n^2), \dots$, and $\Delta(k) = \det W(k)$ its determinant. Let

$$K(t) = \prod_{1 \leq m \leq n} (t - k_m) \quad (16)$$

and let \mathcal{C} be the $n \times n$ matrix $(c_{im})_{1 \leq i, m \leq n}$, where the c_{im} 's are defined by the partial fraction expansions:

$$1 \leq i \leq n \quad \frac{t^{i-1}}{K(t)} = \sum_{1 \leq m \leq n} \frac{c_{im}}{t - k_m} \quad (17)$$

We have the two matrix equations:

$$\mathcal{C} = W(k) \operatorname{diag}(K'(k_1)^{-1}, \dots, K'(k_n)^{-1}) \quad (18a)$$

$$\mathcal{C} \cdot \left(\frac{1}{l_j - k_m} \right)_{1 \leq m, j \leq n} = W(l) \operatorname{diag}(K(l_1)^{-1}, \dots, K(l_n)^{-1}) \quad (18b)$$

This gives the (well-known) identity:

$$\left(\frac{1}{l_j - k_m} \right)_{1 \leq m, j \leq n} = \operatorname{diag}(K'(k_1), \dots, K'(k_n)) W(k)^{-1} W(l) \operatorname{diag}(K(l_1)^{-1}, \dots, K(l_n)^{-1}) \quad (19)$$

We can thus rewrite the determinant we want to compute as:

$$\left| \frac{u_i y_j - v_i x_j}{l_j - k_i} \right|_{1 \leq i, j \leq n} = \prod_m K'(k_m) \prod_j K(l_j)^{-1} \left| (u_i y_j - v_i x_j) (W(k)^{-1} W(l))_{ij} \right|_{n \times n} \quad (20)$$

We shall now make use of (15) with $A = W(k)$ and $B = W(l)$.

$$\begin{aligned} \left| \frac{u_i y_j - v_i x_j}{l_j - k_i} \right|_{1 \leq i, j \leq n} &= \Delta(k)^{-2} \prod_m K'(k_m) \prod_j K(l_j)^{-1} \left| \begin{matrix} W(k)d(u) & W(l)d(x) \\ W(k)d(v) & W(l)d(y) \end{matrix} \right| \\ &= \frac{(-1)^{\frac{n(n-1)}{2}}}{\prod_{i,j} (l_j - k_i)} \left| \begin{matrix} W(k)d(u) & W(l)d(x) \\ W(k)d(v) & W(l)d(y) \end{matrix} \right|_{2n \times 2n} \end{aligned} \quad (21)$$

The sign $(-1)^{n(n-1)/2} = (-1)^{[\frac{n}{2}]}$ is the signature of the permutation which exchanges rows i and $n+i$ for $i = 1, 2, \dots, 2[\frac{n}{2}]$ and transforms the determinant on the right-hand side into $\begin{vmatrix} U_n & X_n \\ V_n & Y_n \end{vmatrix}$. This concludes the proof. \square

3 A and B for spaces with imaginary trivial zeros

Just using the existence of continuous evaluators but not yet (5), we have by a simple argument of orthogonal projection (see [5]):

Proposition 2. *Let H be a Hilbert space of entire functions with continuous evaluators $Z_z: \forall f \in H \ f(z) = (Z_z, f)$. Let $\sigma = (z_1, \dots, z_n)$ be a finite sequence of distinct complex numbers with associated evaluators Z_1, \dots, Z_n , assumed to be linearly independent. Let $H(\sigma)$ be the Hilbert space of entire functions which are complete forms of the elements of H vanishing on σ . The evaluators of $H(\sigma)$ are given by:*

$$\mathcal{K}_z(w) = \frac{\gamma(w)\overline{\gamma(z)}}{G_n} \begin{vmatrix} (Z_1, Z_1) & \dots & (Z_1, Z_n) & (Z_1, Z_z) \\ (Z_2, Z_1) & \dots & (Z_2, Z_n) & (Z_2, Z_z) \\ \vdots & \dots & \vdots & \vdots \\ (Z_w, Z_1) & \dots & (Z_w, Z_n) & (Z_w, Z_z) \end{vmatrix} \quad (22)$$

where $G_n > 0$ is the principal $n \times n$ minor of the matrix.

Recalling the form (5) of the reproducing kernel:

$$(Z_{w_1}, Z_{w_2}) = 2 \frac{\overline{B(w_2)}A(w_1) - \overline{A(w_2)}B(w_1)}{\overline{w_2} - w_1} \quad (23)$$

we see that the choices:

$$1 \leq i \leq n : \quad u_i = A(z_i), v_i = B(z_i), k_i = z_i \quad (24a)$$

$$u_{n+1} = A(w), v_{n+1} = B(w), k_{n+1} = w, \quad (24b)$$

$$1 \leq j \leq n : \quad x_j = \overline{A(z_j)}, y_j = \overline{B(z_j)}, l_j = \overline{z_j}, \quad (24c)$$

$$x_{n+1} = \overline{A(z)}, y_{n+1} = \overline{B(z)}, l_{n+1} = \overline{z} \quad (24d)$$

allow to make use of Theorem 1. This gives:

$$\mathcal{K}_z(w) = \frac{2^{n+1} \gamma(w) \overline{\gamma(z)} (-1)^n \gamma^*(w) \gamma(\overline{z})}{G_n \cdot \prod_{1 \leq i, j \leq n} (\overline{z_j} - z_i) \cdot (\overline{z} - w)} \begin{vmatrix} U_{n,w} & X_{n,\overline{z}} \\ V_{n,w} & Y_{n,\overline{z}} \end{vmatrix}_{(2n+2) \times (2n+2)} \quad (25)$$

with

$$U_{n,w} = \begin{pmatrix} A(z_1) & A(z_2) & \dots & A(z_n) & A(w) \\ z_1 B(z_1) & z_2 B(z_2) & \dots & z_n B(z_n) & w B(w) \\ z_1^2 A(z_1) & z_2^2 A(z_2) & \dots & z_n^2 A(z_n) & w^2 A(w) \\ \vdots & \dots & \dots & \vdots & \vdots \end{pmatrix} \quad (26)$$

$$V_{n,w} = \begin{pmatrix} B(z_1) & B(z_2) & \dots & B(z_n) & B(w) \\ z_1 A(z_1) & z_2 A(z_2) & \dots & z_n A(z_n) & w A(w) \\ z_1^2 B(z_1) & z_2^2 B(z_2) & \dots & z_n^2 B(z_n) & w^2 B(w) \\ \vdots & \dots & \dots & \vdots & \vdots \end{pmatrix} \quad (27)$$

$$X_{n,\overline{z}} = \overline{U_{n,z}} \quad (28)$$

$$Y_{n,\overline{z}} = \overline{V_{n,z}} \quad (29)$$

We shall now make the following hypotheses: (1) the z_i 's are purely imaginary, (2) A is even and B is odd. Then $\overline{A(z_i)} = A(\overline{z_i}) = A(-z_i) = A(z_i)$, and $\overline{B(z_i)} = B(\overline{z_i}) = B(-z_i) = -B(z_i)$. The first n columns of the matrix $U_{n,w}$ are thus real and identical with the first n columns of $X_{n,\overline{z}}$. The first n columns of the matrix $V_{n,w}$ are purely imaginary and thus the opposite of the first n columns of $Y_{n,\overline{z}}$.

In order to compute the determinant $\begin{vmatrix} U_{n,w} & X_{n,\overline{z}} \\ V_{n,w} & Y_{n,\overline{z}} \end{vmatrix}_{(2n+2) \times (2n+2)}$, we subtract, for $1 \leq j \leq n$, column j to column $j + n + 1$. This sets to zero all columns of $X_{n,\overline{z}}$ except its last and multiplies by 2 the n first columns of $Y_{n,\overline{z}}$. We then apply a Laplace expansion using the $(n+1) \times (n+1)$ minors built with the first and last $(n+1)$ rows. If the top minor has both the w and \overline{z} columns, its complementary bottom minor will have two proportional

columns hence vanish. There are thus only two contributions, and taking (various) signs into account we obtain:

$$\begin{vmatrix} U_{n,w} & X_{n,\bar{z}} \\ V_{n,w} & Y_{n,\bar{z}} \end{vmatrix}_{(2n+2) \times (2n+2)} = 2^n \det(U_{n,w}) \det(Y_{n,\bar{z}}) - \det(X_{n,\bar{z}}) (-2)^n \det(V_{n,w}) \quad (30)$$

So:

$$\mathcal{K}_z(w) = \frac{2^{2n+1} \gamma(w) \overline{\gamma(z)} \gamma(\bar{z}) (-1)^n \gamma^*(w)}{G_n \prod_{i,j \leq n} (\bar{z}_j - z_i)} \cdot \frac{\det(U_{n,w}) \overline{\det(V_{n,z})} - \overline{\det(U_{n,z})} (-1)^n \det(V_{n,w})}{\bar{z} - w} \quad (31a)$$

Let us also compute the Gram determinant G_n . The determinantal identity gives:

$$G_n = \frac{2^n}{\prod_{i,j} (\bar{z}_j - z_i)} \begin{vmatrix} U_n & U_n \\ V_n & -V_n \end{vmatrix}_{2n \times 2n} = \frac{2^{2n} (-1)^n}{\prod_{1 \leq i,j \leq n} (\bar{z}_j - z_i)} \det(U_n) \det(V_n) \quad (32)$$

Finally:

$$\mathcal{K}_z(w) = 2 \gamma(w) \gamma^*(w) \overline{\gamma(z) \gamma^*(z)} \frac{\det(U_{n,w}) \overline{\det(V_{n,z})} - \overline{\det(U_{n,z})} (-1)^n \det(V_{n,w})}{\det(U_n) \det(V_n) (\bar{z} - w)} \quad (33)$$

Taking into account that $i^n \det(V_n)$ is real we get:

$$\mathcal{K}_z(w) = 2 \gamma(w) \gamma^*(w) \overline{\gamma(z) \gamma^*(z)} \frac{\det(U_{n,w}) i^n \det(V_{n,z}) - \overline{\det(U_{n,z})} i^n \det(V_{n,w})}{\det(U_n) (-i)^n \det(V_n) (\bar{z} - w)} \quad (34a)$$

$$= 2 \frac{\gamma(w) \gamma^*(w) \overline{\gamma(z) \gamma^*(z)}}{\bar{z} - w} \left(\frac{(-1)^n \det(V_{n,z})}{\det V_n} \frac{\det(U_{n,w})}{\det U_n} - \frac{\overline{\det(U_{n,z})}}{\det U_n} \frac{(-1)^n \det(V_{n,w})}{\det V_n} \right) \quad (34b)$$

The following has been obtained:

Theorem 3. *Let H be a Hilbert space of entire functions with reproducing kernel $Z_z(w) = 2 \frac{\overline{B(z)} A(w) - \overline{A(z)} B(w)}{\bar{z} - w}$, where the entire functions A and B are real on the real line and respectively even and odd. Let $\sigma = (z_i)_{1 \leq i \leq n}$ be a finite sequence of distinct purely imaginary numbers. We assume that the associated evaluators are linearly independent, and also that $z_i + z_j \neq 0$ for all i, j . Let $H(\sigma)$ be the Hilbert space of the functions $\gamma(z) f(z)$, where $\gamma(z) = \prod_i \frac{1}{z - z_i}$ and f is in H with $f(z) = 0$ for $z \in \sigma$. Let*

$$\mathcal{A}_\sigma(w) = \frac{(-1)^{\frac{n(n-1)}{2}} \gamma(w) \gamma^*(w)}{\det U_n} \begin{vmatrix} A(z_1) & A(z_2) & \dots & A(z_n) & A(w) \\ z_1 B(z_1) & z_2 B(z_2) & \dots & z_n B(z_n) & w B(w) \\ z_1^2 A(z_1) & z_2^2 A(z_2) & \dots & z_n^2 A(z_n) & w^2 A(w) \\ \vdots & \dots & \dots & \dots & \vdots \end{vmatrix}_{(n+1) \times (n+1)} \quad (35a)$$

$$\mathcal{B}_\sigma(w) = \frac{(-1)^{\frac{n(n+1)}{2}} \gamma(w) \gamma^*(w)}{\det V_n} \begin{vmatrix} B(z_1) & B(z_2) & \dots & B(z_n) & B(w) \\ z_1 A(z_1) & z_2 A(z_2) & \dots & z_n A(z_n) & w A(w) \\ z_1^2 B(z_1) & z_2^2 B(z_2) & \dots & z_n^2 B(z_n) & w^2 B(w) \\ \vdots & \dots & \dots & \dots & \vdots \end{vmatrix}_{(n+1) \times (n+1)} \quad (35b)$$

where the denominators $\det U_n$ and $\det V_n$ are the principal $n \times n$ minors of the numerators. The space $H(\sigma)$ has evaluators \mathcal{K}_z satisfying the formula:

$$\mathcal{K}_z(w) = (\mathcal{K}_w, \mathcal{K}_z) = 2 \frac{\overline{\mathcal{B}_\sigma(z)} \mathcal{A}_\sigma(w) - \overline{\mathcal{A}_\sigma(z)} \mathcal{B}_\sigma(w)}{\bar{z} - w} \quad (36)$$

The functions \mathcal{A}_σ and \mathcal{B}_σ are entire, real on the real line, \mathcal{A}_σ is even and \mathcal{B}_σ is odd.

Remark 1. The additional $(-1)^{\frac{n(n-1)}{2}}$ is to make $\mathcal{A}_\sigma(it) > 0$ and $-i\mathcal{B}_\sigma(it) > 0$ for $t > 0$, at least in the case of the Paley-Wiener spaces (this sign is easily determined from the asymptotics as $t \rightarrow +\infty$; let us also mention that $-i\mathcal{B}_\sigma(it)\mathcal{A}_\sigma(it) > 0$ for $t > 0$, from (36) and if $H(\sigma) \neq \{0\}$). We observe that if the initial A and B verify the normalization $\frac{-iB(it)}{A(it)} \rightarrow_{t \rightarrow +\infty} 1$ then the new \mathcal{A}_σ and \mathcal{B}_σ also. This normalization has proven to be more natural in this and other investigations, than other normalizations such as, for example, $A(0) = 1$ (when possible).

Remark 2. A formula of the type (36) for evaluators in a Hilbert space of entire functions is guaranteed by the axiomatic framework of [2]. The passage from an H to an $H(\sigma)$ is compatible to these axioms (cf. [5]), hence existence of an $\mathcal{E}_\sigma = \mathcal{A}_\sigma - i\mathcal{B}_\sigma$ function was known in advance. Determination of a suitable \mathcal{E}_σ , without any of the restrictive hypotheses made here, is achieved in [5] with another method. Confrontation of the results thus establishes some interesting identities.

Let us record a special case of the computation (32) of the Gram determinant G_n , using the notation $W(f_1, \dots, f_n)$ for Wronskian determinants $\det(f_j^{(i)})_{1 \leq i, j \leq n}$ (derivatives with respect to x):

Proposition 4. *The following identity holds:*

$$\left| \frac{\text{sh}((\kappa_i + \kappa_j)x)}{\kappa_i + \kappa_j} \right|_{1 \leq i, j \leq n} = \frac{W(\text{ch}(\kappa_1 x), \dots, \text{ch}(\kappa_n x)) \cdot W(\text{sh}(\kappa_1 x), \dots, \text{sh}(\kappa_n x))}{\prod_{1 \leq i \leq n} \kappa_i \prod_{1 \leq i < j \leq n} (\kappa_i + \kappa_j)^2} \quad (37)$$

Proof. Let $u_i = \text{ch}(\kappa_i x)$, $v_i = \text{sh}(\kappa_i x)$, $k_i = \kappa_i$, $x_j = \text{ch}(\kappa_j x) = u_j$, $y_j = -\text{sh}(\kappa_j x) = -v_j$, $l_j = -\kappa_j = -k_j$. With these choices:

$$\frac{\text{sh}((\kappa_i + \kappa_j)x)}{\kappa_i + \kappa_j} = \frac{u_i y_j - v_i x_j}{l_j - k_i} \quad (38)$$

By Theorem 1:

$$\det_{1 \leq i, j \leq n} \left(\frac{u_i y_j - v_i x_j}{l_j - k_i} \right) = \frac{1}{\prod_{i,j} (l_j - k_i)} \begin{vmatrix} U_n & X_n \\ V_n & Y_n \end{vmatrix}_{2n \times 2n} = \frac{(-1)^n (-2)^n}{\prod_{i,j} (\kappa_i + \kappa_j)} \det(U_n) \det(V_n) \quad (39)$$

where we used $U_n = X_n$, $V_n = -Y_n$. As $\det U_n = W(\text{ch}(\kappa_1 x), \dots, \text{ch}(\kappa_n x))$ and $\det V_n = W(\text{sh}(\kappa_1 x), \dots, \text{sh}(\kappa_n x))$, this completes the proof. \square

4 Crum formulas for Darboux transformations of Krein systems

All derivatives in this chapter will be with respect to a variable x . We are interested in differential systems of the Krein type:

$$(S) \begin{cases} a' - \mu a &= -kb \\ b' + \mu b &= +ka \end{cases} \quad (40)$$

Krein uses systems of this type in particular in his approach [10] to Inverse Scattering Theory and in his continuous analogues to topics of Orthogonal Polynomial Theory [11]. The system couples two Schrödinger equations:

$$-a'' + V^+ a = k^2 a \quad \text{with } V^+ = \mu^2 + \mu' \quad (41a)$$

$$-b'' + V^- b = k^2 b \quad \text{with } V^- = \mu^2 - \mu' \quad (41b)$$

It proves quite convenient to introduce the notion of a tau-function, which is a function such that:

$$\mu^2 = -(\log \tau)'' \quad (42)$$

We shall also use the notation $\lambda = (\log \tau)'$, so that $\mu^2 = -\lambda'$.

The well-known Darboux transformation [7, §6] transforms the solutions of a Schrödinger equation $-f'' + V f = E f$ into solutions of another one, and the formulas of Crum [6] give Wronskian expressions for both solutions and potentials after successive such Darboux transformations. In this chapter we introduce a notion of “simultaneous” or “linked” such transformations which act at the level of the Krein system (40). This provides a kind of refinement to the formula of Crum, the change of the two potentials being lifted to the change of the “tau” function. We did not find in the literature the results we prove here, but it is so extensive that we may have missed some important contributions.

We make use also of couples (α, β) of the type $(a, -ib)$. Hence we also consider the differential systems:

$$(T) \begin{cases} \alpha' - \mu \alpha &= +\kappa \beta \\ \beta' + \mu \beta &= +\kappa \alpha \end{cases} \quad (43)$$

It corresponds to (S) (40) via $k = i\kappa$, $a = \alpha$, $b = i\beta$. The Schrödinger equations become:

$$\alpha'' = (\kappa^2 + V^+)\alpha \quad (44a)$$

$$\beta'' = (\kappa^2 + V^-)\beta. \quad (44b)$$

Theorem 5. *Let $\kappa \in \mathbb{C}$ and let (α, β) be a solution of the differential system (T) (43), with neither α nor β identically zero. The simultaneous Darboux transformations:*

$$a \rightarrow a_1 = a' - \frac{\alpha'}{\alpha}a \quad (45a)$$

$$b \rightarrow b_1 = b' - \frac{\beta'}{\beta}b \quad (45b)$$

transform any solution (a, b, k) of the differential system (S) (40) into a solution (a_1, b_1, k) of a transformed system:

$$(S_1) \begin{cases} a'_1 - \mu_1 a_1 &= -kb_1 \\ b'_1 + \mu_1 b_1 &= +ka_1 \end{cases} \quad (46a)$$

where the new coefficient μ_1 is

$$\mu_1 = \mu - \frac{d}{dx} \log \frac{\alpha}{\beta} \quad (47)$$

If $\mu^2 = -(\log \tau)''$ then $\mu_1^2 = -(\log \tau_1)''$ with

$$\tau_1 = \tau \alpha \beta \quad (48)$$

Proof. From $\alpha a_1 = \left| \frac{\alpha}{\alpha'} \frac{a}{a'} \right|$, we get $(\alpha a_1)' = \left| \frac{\alpha}{\alpha'} \frac{a}{a'} \right|' = \left| (V^+ + \kappa^2)_\alpha (V^+ - k^2)_a \right| = -(k^2 + \kappa^2)\alpha a$, which we rewrite as

$$a'_1 + \frac{\alpha'}{\alpha}a_1 = -(k^2 + \kappa^2)a = -k(b' + \mu b) - \kappa^2 a \quad (49)$$

Further

$$\alpha a_1 = \alpha(-kb + \mu a) - (\kappa\beta + \mu\alpha)a = -k\alpha b - \kappa\beta a \quad (50)$$

Eliminating a gives:

$$a'_1 + \frac{\alpha'}{\alpha}a_1 - \kappa \frac{\alpha}{\beta}a_1 = -k(b' + \mu b) + k\kappa \frac{\alpha}{\beta}b \quad (51)$$

Using $\kappa \frac{\alpha}{\beta} = \frac{\beta'}{\beta} + \mu$:

$$a'_1 + \left(\frac{\alpha'}{\alpha} - \frac{\beta'}{\beta} - \mu \right) a_1 = -k(b' - \frac{\beta'}{\beta}b) \quad (52)$$

With the definitions $\mu_1 = \mu - \frac{\alpha'}{\alpha} + \frac{\beta'}{\beta}$ and $b_1 = b' - \frac{\beta'}{\beta}b$ this gives indeed:

$$a'_1 - \mu_1 a_1 = -kb_1 \quad (53)$$

From $\beta b_1 = \left| \begin{smallmatrix} \beta & b \\ \beta' & b' \end{smallmatrix} \right|$, we get $(\beta b_1)' = \left| \begin{smallmatrix} \beta & b \\ \beta'' & b'' \end{smallmatrix} \right| = \left| \begin{smallmatrix} \beta & b \\ (V^- + \kappa^2)\beta & (V^- - \kappa^2)b \end{smallmatrix} \right| = -(k^2 + \kappa^2)b\beta$, which gives:

$$b'_1 + \frac{\beta'}{\beta}b_1 = -(k^2 + \kappa^2)b = k(a' - \mu a) - \kappa^2 b \quad (54)$$

On the other hand

$$\beta b_1 = \beta(ka - \mu b) - (\kappa\alpha - \mu\beta)b = k\beta a - \kappa\alpha b \quad (55)$$

Eliminating b gives:

$$b'_1 + \frac{\beta'}{\beta}b_1 - \kappa\frac{\beta}{\alpha}b_1 = k(a' - \mu a) - k\kappa\frac{\beta}{\alpha}a \quad (56)$$

Using $\kappa\frac{\beta}{\alpha} = \frac{\alpha'}{\alpha} - \mu$ finally leads to:

$$b'_1 + \left(\frac{\beta'}{\beta} - \frac{\alpha'}{\alpha} + \mu\right)b_1 = k(a' - \frac{\alpha'}{\alpha}a) \implies b'_1 + \mu_1 b_1 = +ka_1 \quad (57)$$

Let $\lambda_1 = \lambda + \frac{\alpha'}{\alpha} + \frac{\beta'}{\beta} = (\log \tau\alpha\beta)'$. We must also verify $\mu_1^2 = -\lambda'_1$.

$$\lambda'_1 = \lambda' + (V^+ + \kappa^2) - \left(\frac{\alpha'}{\alpha}\right)^2 + (V^- + \kappa^2) - \left(\frac{\beta'}{\beta}\right)^2 = \mu^2 + 2\kappa^2 - \left(\frac{\alpha'}{\alpha}\right)^2 - \left(\frac{\beta'}{\beta}\right)^2 \quad (58)$$

$$\kappa^2 = \frac{(\alpha' - \mu\alpha)(\beta' + \mu\beta)}{\alpha\beta} = \frac{\alpha'}{\alpha}\frac{\beta'}{\beta} - \mu\frac{\beta'}{\beta} + \mu\frac{\alpha'}{\alpha} - \mu^2 \quad (59)$$

$$\lambda'_1 = -\mu^2 + 2\frac{\alpha'}{\alpha}\frac{\beta'}{\beta} - 2\mu\left(\frac{\beta'}{\beta} - \frac{\alpha'}{\alpha}\right) - \left(\frac{\alpha'}{\alpha}\right)^2 - \left(\frac{\beta'}{\beta}\right)^2 = -\mu_1^2 \quad (60)$$

□

Remark 3. A solution (a, b, k) of system (S) (40) corresponds to a solution $(\alpha, \beta, \kappa) = (a, -ib, -ik)$ of system (T) (43), and to a solution (α, β, κ) of (T) we can switch to the solution $(\alpha, i\beta, i\kappa)$ of (S) , having the same logarithmic derivatives with respect to x . Hence it is just a matter of arbitrary choice to consider the Darboux transformations to be associated to a specific solution of (T) rather than to a specific solution of (S) . Moreover, the same Darboux transformations (45a), (45b) which are associated to a given (α, β, κ) but now applied to a triple (γ, δ, ξ) , solution of (T) , produces a solution of the transformed system

$$(T_1) \begin{cases} \gamma'_1 - \mu_1\gamma_1 &= \xi\delta_1 \\ \delta'_1 + \mu_1\delta_1 &= \xi\gamma_1 \end{cases} \quad (61a)$$

of type (T) associated to the new coefficient μ_1 .

Theorem 6. *Let there be given n triples $(\alpha_j, \beta_j, \kappa_j)$, solutions of the differential system (T) (43). We assume that $\alpha_1, \dots, \alpha_n$ are linearly independent, and β_1, \dots, β_n also. To each solution (a, b, k) of the system*

$$(S) \begin{cases} a' - \mu a &= -kb \\ b' + \mu b &= +ka \end{cases} \quad (62)$$

we associate

$$a_n = \frac{W(\alpha_1, \dots, \alpha_n, a)}{W(\alpha_1, \dots, \alpha_n)} \quad (63a)$$

$$b_n = \frac{W(\beta_1, \dots, \beta_n, b)}{W(\beta_1, \dots, \beta_n)} \quad (63b)$$

Going from (a, b) to (a_n, b_n) is the result of the n successive simultaneous Darboux transformations (45a) and (45b) associated to $(\alpha_1, \beta_1), \dots, (\alpha_n, \beta_n)$ (themselves transformed along the way). There holds:

$$(S_n) \begin{cases} a'_n - \mu_n a_n &= -kb_n \\ b'_n + \mu_n b_n &= +ka_n \end{cases} \quad (64)$$

where the coefficient μ_n is given by:

$$\mu_n = \mu - \frac{d}{dx} \log \frac{W(\alpha_1, \dots, \alpha_n)}{W(\beta_1, \dots, \beta_n)} \quad (65)$$

If furthermore one chooses a tau-function such that $\mu^2 = -(\log \tau)''$ then

$$\mu_n^2 = -\frac{d^2}{dx^2} \log \tau_n \quad (66)$$

where

$$\tau_n = \tau \cdot W(\alpha_1, \dots, \alpha_n) W(\beta_1, \dots, \beta_n) \quad (67)$$

Proof. Let us consider first the simultaneous Darboux transformations of system (S) (40) and of its partner (T) (43), defined by (α_1, β_1) . Let us write in particular $(\alpha_2^{(1)}, \beta_2^{(1)})$ for the transform of the couple (α_2, β_2) . We then apply the associated Darboux transformations to (S_1) giving rise to (S_2) . The couple (α_3, β_3) is transformed into a solution $(\alpha_3^{(2)}, \beta_3^{(2)})$ of partner (T_2) . Etc... Although we speak of transformed systems to keep track of the coupling, each of the associated Schrödinger equation $-f'' + V^\pm f = k^2 f$ is transformed by $f \mapsto f' - \frac{g'}{g} f$ where g is a solution of $-g'' + V^\pm g = -\kappa^2 g$, hence independently of what happens to the other equation $-\phi'' + V^\mp \phi = k^2 \phi$. One part of the Theorem of Crum [6] (which we do not reprove here) tells us that if we apply n successive Darboux transformations $f \mapsto f' - \frac{g'}{g} f$, first by g_1 , then by the transformed g_2 , then by the transformed g_3 , ... the final action can be written directly as:

$$f \mapsto \frac{W(g_1, \dots, g_n, f)}{W(g_1, \dots, g_n)} \quad (68)$$

Hence definitions (63a) and (63b) of a_n and b_n can be viewed as the final result of the n successive simultaneous Darboux transformations. Theorem 5 tells us how μ changes when system (S) is transformed once, hence iterative use of the Theorem gives a formula for μ_n involving in fact telescopic products of quotients of Wronskians, hence equation (65). Moreover if a tau function is initially chosen with $-(\log \tau)'' = \mu^2$, Theorem 5 can again be applied iteratively, leading to a function τ_n given by (67), and verifying $-(\log \tau_n)'' = \mu_n^2$. \square

Let us take note that $\mu_n^2 + \mu'_n = -(\log \tau)'' - (\log W(\alpha_1, \dots, \alpha_n))'' - (\log W(\beta_1, \dots, \beta_n))'' + \mu' - \frac{d^2}{dx^2} \log \frac{W(\alpha_1, \dots, \alpha_n)}{W(\beta_1, \dots, \beta_n)} = \mu^2 + \mu' - 2 \frac{d^2}{dx^2} \log W(\alpha_1, \dots, \alpha_n)$. And similarly $\mu_n^2 - \mu'_n = \mu^2 - \mu' - 2 \frac{d^2}{dx^2} \log W(\beta_1, \dots, \beta_n)$. Thus:

Corollary 7. *Using the notations of Theorem 6, there holds*

$$-a_n'' + V_n^+ a_n = k^2 a_n \quad (69a)$$

$$-b_n'' + V_n^- b_n = k^2 b_n \quad (69b)$$

with

$$V_n^+ = V^+ - 2 \frac{d^2}{dx^2} \log W(\alpha_1, \dots, \alpha_n) \quad (70a)$$

$$V_n^- = V^- - 2 \frac{d^2}{dx^2} \log W(\beta_1, \dots, \beta_n) \quad (70b)$$

These formulas are the part of Crum's Theorem [6] regarding the effect of successive Darboux transformations on the potentials of Schrödinger equations.

5 Modification of mu-functions by trivial zeros

We are interested in Hilbert spaces H_x of entire functions in the sense of [2], whose reproducing kernels are given by formula (5), where the functions A ($= A_x$) and B ($= B_x$) are real valued on the real line, respectively even and odd, and obey a first order differential system with respect to x of the Krein type [10], with a real valued coefficient function $\mu(x)$:

$$\frac{d}{dx} A_x(w) - \mu(x) A_x(w) = -w B_x(w) \quad (71a)$$

$$\frac{d}{dx} B_x(w) + \mu(x) B_x(w) = w A_x(w) \quad (71b)$$

Remark 4. More general integral equations play the important general structural role in [2]. We have found that the above restricted type arises naturally in the study of some specific instances of Hilbert spaces of entire functions [3]. It turns out to be well adapted

to the present study of the classical Paley-Wiener spaces modified by adding trivial zeros on the imaginary axis. If we remove the restriction for the zeros to lie on the imaginary axis, the functions A_x and B_x real on the real line will (generally speaking) cease to be respectively even and odd and they obey the more general type of Krein system from [11] which has both the real and imaginary parts of a complex valued μ -function as coefficients.

We want to combine Theorem 3 and Theorem 6. We will suppose that the functions A_x are even, the functions B_x odd, and the trivial zeros z_i , $1 \leq i \leq n$, are purely imaginary and verify $z_i \neq \pm z_j$ for all i, j .

From (71a) and (71b):

$$\left[\left(\frac{d}{dx} + \mu \right) \left(\frac{d}{dx} - \mu \right) \right]^{2p} A_x = (-1)^p w^{2p} A_x \quad (72a)$$

$$\left(\frac{d}{dx} - \mu \right) \left[\left(\frac{d}{dx} + \mu \right) \left(\frac{d}{dx} - \mu \right) \right]^{2p} A_x = (-1)^{p+1} w^{2p+1} B_x \quad (72b)$$

By recurrence the left side of (72a) (resp. (72b)) is $(\frac{d}{dx})^{(2p)} A_x$ (resp. $(\frac{d}{dx})^{(2p+1)} A_x$) up to a finite linear combination of lower derivatives of A_x with coefficients being function of x (independent of w). Hence, for $n = 2m$:

$$W(A_x(z_1), \dots, A_x(z_n), A_x(w)) = \begin{vmatrix} A_x(z_1) & \dots & A_x(z_n) & A_x(w) \\ z_1 B_x(z_1) & \dots & z_n B_x(z_n) & w B_x(w) \\ z_1^2 A_x(z_1) & \dots & z_n^2 A_x(z_n) & w^2 A_x(w) \\ \vdots & \dots & \dots & \vdots \\ z_1^{2m} A_x(z_1) & \dots & \dots & w^{2m} A_x(w) \end{vmatrix}_{(2m+1) \times (2m+1)} \quad (73)$$

and for $n = 2m + 1$:

$$W(A_x(z_1), \dots, A_x(z_n), A_x(w)) = (-1)^{m+1} \begin{vmatrix} A_x(z_1) & \dots & A_x(z_n) & A_x(w) \\ z_1 B_x(z_1) & \dots & z_n B_x(z_n) & w B_x(w) \\ z_1^2 A_x(z_1) & \dots & z_n^2 A_x(z_n) & w^2 A_x(w) \\ \vdots & \dots & \vdots & \vdots \\ z_1^{2m} A_x(z_1) & \dots & \dots & w^{2m} A_x(w) \\ z_1^{2m+1} B_x(z_1) & \dots & \dots & w^{2m+1} B_x(w) \end{vmatrix}_{(2m+2) \times (2m+2)} \quad (74)$$

If we divide the Wronskians (constructed with derivations with respect to x) and the $(n+1) \times (n+1)$ determinants at the right by their respective $n \times n$ principal minors, the resulting fractions will thus coincide up to $(-1)^m$ for $n = 2m$ and $(-1)^{m+1}$ for $n = 2m + 1$, hence in both cases up to $(-1)^{\frac{1}{2}n(n+1)}$. We can thus rewrite the function \mathcal{A}_σ of Theorem 3 as:

$$\mathcal{A}_\sigma(w) = (-1)^n \gamma(w) \gamma^*(w) \frac{W(A_x(z_1), A_x(z_2), \dots, A_x(z_n), A_x(w))}{W(A_x(z_1), A_x(z_2), \dots, A_x(z_n))} \quad (75)$$

In the same manner

$$\left[\left(\frac{d}{dx} - \mu \right) \left(\frac{d}{dx} + \mu \right) \right]^{2p} B_x = (-1)^p w^{2p} B_x \quad (76a)$$

$$\left(\frac{d}{dx} + \mu \right) \left[\left(\frac{d}{dx} - \mu \right) \left(\frac{d}{dx} + \mu \right) \right]^{2p} B_x = (-1)^p w^{2p+1} A_x \quad (76b)$$

By recurrence the left side of (76a) (resp. (76b)) is $(\frac{d}{dx})^{(2p)} B_x$ (resp. $(\frac{d}{dx})^{(2p+1)} B_x$) up to a finite linear combination of lower derivatives of B_x with coefficients being function of x (independent of w). Hence, for $n = 2m$:

$$W(B_x(z_1), \dots, B_x(z_n), B_x(w)) = (-1)^m \begin{vmatrix} B_x(z_1) & \dots & B_x(z_n) & B_x(w) \\ z_1 A_x(z_1) & \dots & z_n A_x(z_n) & w A_x(w) \\ z_1^2 B_x(z_1) & \dots & z_n^2 B_x(z_n) & w^2 B_x(w) \\ \vdots & \dots & \dots & \vdots \\ z_1^{2m} B_x(z_1) & \dots & \dots & w^{2m} B_x(w) \end{vmatrix}_{(2m+1) \times (2m+1)} \quad (77)$$

and for $n = 2m + 1$:

$$W(B_x(z_1), \dots, B_x(z_n), B_x(w)) = \begin{vmatrix} B_x(z_1) & \dots & B_x(z_n) & B_x(w) \\ z_1 A_x(z_1) & \dots & z_n A_x(z_n) & w A_x(w) \\ z_1^2 B_x(z_1) & \dots & z_n^2 B_x(z_n) & w^2 B_x(w) \\ \vdots & \dots & \vdots & \vdots \\ z_1^{2m} B_x(z_1) & \dots & \dots & w^{2m} B_x(w) \\ z_1^{2m+1} A_x(z_1) & \dots & \dots & w^{2m+1} A_x(w) \end{vmatrix}_{(2m+2) \times (2m+2)} \quad (78)$$

If we divide the Wronskians and the determinants at the right by their respective $n \times n$ principal minors, the results will coincide up to $(-1)^m$ for $n = 2m$ and $(-1)^m$ for $n = 2m+1$, hence in both cases up to $(-1)^{\frac{1}{2}n(n-1)}$. We can rewrite the function \mathcal{B}_σ of Theorem 3 as:

$$\mathcal{B}_\sigma(w) = (-1)^n \gamma(w) \gamma^*(w) \frac{W(B_x(z_1), B_x(z_2), \dots, B_x(z_n), B_x(w))}{W(B_x(z_1), B_x(z_2), \dots, B_x(z_n))} \quad (79)$$

where the Wronskians are constructed with derivations with respect to x .

Taking into account that $(-1)^n \gamma^*(w) = \gamma(-w)$ we can thus sum up these computations in the following:

Theorem 8. *Let there be given Hilbert spaces H_x of entire functions, with functions A_x (even, real on the real line) and B_x (odd, real on the real line) computing the evaluators in H_x by formula (5), and whose variations with respect to the parameter x are given by:*

$$\frac{d}{dx} A_x(w) - \mu(x) A_x(w) = -w B_x(w) \quad (80a)$$

$$\frac{d}{dx} B_x(w) + \mu(x) B_x(w) = w A_x(w) \quad (80b)$$

Let $\sigma = (z_1, \dots, z_n)$ be a finite sequence of purely imaginary numbers (the associated evaluators in H_x being supposed linearly independent) with $z_i \neq \pm z_j$ for $1 \leq i, j \leq n$ and let $H_x(\sigma)$ be the Hilbert space H_x modified by σ . Its evaluators \mathcal{K}_z are given by:

$$\mathcal{K}_z(w) = (\mathcal{K}_w, \mathcal{K}_z) = 2 \frac{\overline{\mathcal{B}_\sigma(z)} \mathcal{A}_\sigma(w) - \overline{\mathcal{A}_\sigma(z)} \mathcal{B}_\sigma(w)}{\bar{z} - w} \quad (81)$$

with

$$\mathcal{A}_{x,\sigma}(w) = \gamma(w)\gamma(-w) \frac{W(A_x(z_1), A_x(z_2), \dots, A_x(z_n), A_x(w))}{W(A_x(z_1), A_x(z_2), \dots, A_x(z_n))} \quad (82a)$$

$$\mathcal{B}_{x,\sigma}(w) = \gamma(w)\gamma(-w) \frac{W(B_x(z_1), B_x(z_2), \dots, B_x(z_n), B_x(w))}{W(B_x(z_1), B_x(z_2), \dots, B_x(z_n))} \quad (82b)$$

where the Wronskians involve derivatives with respect to the variable x . The entire functions $\mathcal{A}_{x,\sigma}$ and $\mathcal{B}_{x,\sigma}$ are real on the real line, and respectively even and odd.

Taking into account Theorem 6 we thus learn that:

Theorem 9. Let there be given Hilbert spaces H_x of entire functions, functions A_x and B_x , imaginary numbers z_1, z_2, \dots verifying the hypotheses of Theorem 8. Let $H_x(n) = H_x(z_1, \dots, z_n)$ and let the functions $\mathcal{A}_{x,n}$ and $\mathcal{B}_{x,n}$ computing the reproducing kernel in $H_x(n)$ be provided by Theorem 3. They are obtained by successive transformations (essentially) of Darboux type:

$$\mathcal{A}_{x,n+1}(w) = \frac{1}{z_{n+1}^2 - w^2} \left(\frac{d}{dx} \mathcal{A}_{x,n}(w) - \frac{\frac{d}{dx} \mathcal{A}_{x,n}(z_{n+1})}{\mathcal{A}_{x,n}(z_{n+1})} \mathcal{A}_{x,n}(w) \right) \quad (83a)$$

$$\mathcal{B}_{x,n+1}(w) = \frac{1}{z_{n+1}^2 - w^2} \left(\frac{d}{dx} \mathcal{B}_{x,n}(w) - \frac{\frac{d}{dx} \mathcal{B}_{x,n}(z_{n+1})}{\mathcal{B}_{x,n}(z_{n+1})} \mathcal{B}_{x,n}(w) \right) \quad (83b)$$

and verify the equations

$$\frac{d}{dx} \mathcal{A}_{x,n}(w) - \mu_n(x) \mathcal{A}_{x,n}(w) = -w \mathcal{B}_{x,n}(w) \quad (84a)$$

$$\frac{d}{dx} \mathcal{B}_{x,n}(w) + \mu_n(x) \mathcal{B}_{x,n}(w) = +w \mathcal{A}_{x,n}(w) \quad (84b)$$

with

$$\mu_n = \mu - \frac{d}{dx} \log \frac{W(A_x(z_1), \dots, A_x(z_n))}{W(B_x(z_1), \dots, B_x(z_n))} \quad (85)$$

If a function τ is chosen with $\mu^2 = -\frac{d^2}{dx^2} \log \tau$ then $\mu_n^2 = -\frac{d^2}{dx^2} \log \tau_n$ with

$$\tau_n = \tau \cdot W(A_x(z_1), \dots, A_x(z_n)) W(-iB_x(z_1), \dots, -iB_x(z_n)) \quad (86)$$

In the following, the index x shall be dropped from the notations. Combining (83a) with (84a) we obtain:

$$\begin{aligned}
(z_{n+1}^2 - w^2)\mathcal{A}_{n+1}(w) &= \mu_n \mathcal{A}_n(w) - w\mathcal{B}_n(w) - (\mu_n - z_{n+1} \frac{\mathcal{B}_n(z_{n+1})}{\mathcal{A}_n(z_{n+1})})\mathcal{A}_n(w) \\
&= -(z_{n+1} + w)\mathcal{B}_n(w) + \frac{z_{n+1}}{\mathcal{A}_n(z_{n+1})} (\mathcal{B}_n(z_{n+1})\mathcal{A}_n(w) + \mathcal{A}_n(z_{n+1})\mathcal{B}_n(w)) \\
&= -(z_{n+1} + w)\mathcal{B}_n(w) + \frac{z_{n+1}}{2\mathcal{A}_n(z_{n+1})} (z_{n+1} + w)K^n(z_{n+1}, w)
\end{aligned} \tag{87}$$

We have written $K^n(z, w) = K_z^n(w)$ for the evaluator in $H(n) = H(z_1, \dots, z_n)$. Combining (83b) with (84b) gives:

$$\begin{aligned}
(z_{n+1}^2 - w^2)\mathcal{B}_{n+1}(w) &= -\mu_n \mathcal{B}_n(w) + w\mathcal{A}_n(w) - (-\mu_n + z_{n+1} \frac{\mathcal{A}_n(z_{n+1})}{\mathcal{B}_n(z_{n+1})})\mathcal{B}_n(w) \\
&= (z_{n+1} + w)\mathcal{A}_n(w) - \frac{z_{n+1}}{\mathcal{B}_n(z_{n+1})} (\mathcal{A}_n(z_{n+1})\mathcal{B}_n(w) + \mathcal{B}_n(z_{n+1})\mathcal{A}_n(w)) \\
&= (z_{n+1} + w)\mathcal{A}_n(w) - \frac{z_{n+1}}{2\mathcal{B}_n(z_{n+1})} (z_{n+1} + w)K^n(z_{n+1}, w)
\end{aligned} \tag{88}$$

We thus have the identities:

Theorem 10. *Let $H = H_x$, \mathcal{A}_n , \mathcal{B}_n , for $n \geq 1$ be as in Theorem 9. There holds*

$$(w - z_{n+1})\mathcal{A}_{n+1}(w) = \mathcal{B}_n(w) - \frac{z_{n+1}}{2\mathcal{A}_n(z_{n+1})} K^n(z_{n+1}, w) \tag{89a}$$

$$(w - z_{n+1})\mathcal{B}_{n+1}(w) = -\mathcal{A}_n(w) + \frac{z_{n+1}}{2\mathcal{B}_n(z_{n+1})} K^n(z_{n+1}, w) \tag{89b}$$

where $K^n(z, w) = K_z^n(w)$ is the reproducing kernel in $H(n) = H(z_1, \dots, z_n)$.

From formula (22) in Theorem 2 we know that $\prod_{1 \leq i \leq n} (w - z_i) \cdot K^n(z_{n+1}, w)$ is a linear combination of the initial evaluators $Z_i(w)$ ($= Z_{z_i}(w)$), $1 \leq i \leq n+1$. Hence by induction we obtain the following:

Theorem 11. *Let $H = H_x$, \mathcal{A}_n , \mathcal{B}_n , for $n \geq 1$ be as in Theorem 9. Let $\mathcal{E}_n = \mathcal{A}_n - i\mathcal{B}_n$ and $\mathcal{F}_n = \mathcal{E}_n^* = \mathcal{A}_n + i\mathcal{B}_n$. The function $(-i)^n \prod_{1 \leq i \leq n} (w - z_i) \cdot \mathcal{E}_n(w)$ differs from the initial $E = A - iB$ function by a finite linear combination of the initial evaluators $Z_i(w)$, $1 \leq i \leq n$. Also the function $i^n \prod_{1 \leq i \leq n} (w - z_i) \cdot \mathcal{F}_n(w)$ ($\mathcal{F}_n = \mathcal{E}_n^*$) differs from the initial $F = A + iB$ function by a finite linear combination of the initial evaluators $Z_i(w)$, $1 \leq i \leq n$.*

Remark 5. Let us note that this characterizes uniquely the \mathcal{E}_n (and \mathcal{F}_n) provided by theorem 8, as the unknown linear combinations of (linearly independent) evaluators will be constrained by their values at the z_i 's. This theorem for the transition from H to $H(\sigma)$ holds with much greater generality than achieved here (see the companion article [5]): it

suffices for H to verify the axioms of [2]. Thus, reverting the steps we could have started from the results proven in [5] and, under the additional hypotheses made here (existence of a parameter x and of a differential system of Krein type, imaginary trivial zeros, ...), obtain the Darboux transformations ((83a) and (83b) in Theorem 9) and later the Wronskian formulas (Theorem 8) as corollaries.

6 Non-linear equations for Paley-Wiener spaces with trivial zeros

On the basis of Theorem 11 it is convenient to work with the “incomplete” forms of the various objects encountered. As the main results of this chapter are for the classical Paley-Wiener spaces PW_x , we will from the start assume $H = PW_x$. We consider its modification $H(\sigma)$ by finitely many “trivial” distinct zeros $\sigma = (z_1, \dots, z_n)$ (the associated evaluators in H are always linearly independent). Let $\gamma(w) = \prod_{1 \leq j \leq n} \frac{1}{w - z_j}$ be the corresponding gamma factor. We define the incomplete version $K^\sigma(z, w)$ of the reproducing kernel $\mathcal{K}(z, w)$ in $H(\sigma)$ via the relation

$$\mathcal{K}(z, w) = \mathcal{K}_z(w) = (\mathcal{K}_w, \mathcal{K}_z) = \gamma(w) \overline{\gamma(z)} K^\sigma(z, w) \quad (90)$$

Theorem 2 is the statement that $K^\sigma(z, w)$ is the unique entire function of w which vanishes at z_1, \dots, z_n and differs additively from the initial evaluator $Z(z, w)$ by a finite linear combination of the initial evaluators $Z(z_1, w), \dots, Z(z_n, w)$.

Let us now consider the functions \mathcal{E}_σ and \mathcal{F}_σ characterized as in Theorem 11. We consider their incomplete versions, up to a factor i^n :

$$\mathcal{E}_\sigma(w) = i^n \gamma(w) E_\sigma(w) \quad \mathcal{F}_\sigma(w) = i^n \gamma(w) F_\sigma(w) \quad (91)$$

Of course, there does not hold (for $n \geq 1$) $F_\sigma = E_\sigma^*$ (this last function has its trivial zeros not at the z_i 's but at the \bar{z}_i 's). The formula for the incomplete reproducing kernel is

$$K^\sigma(z, w) = \frac{\overline{E_\sigma(z)} E_\sigma(w) - \overline{F_\sigma(z)} F_\sigma(w)}{i(\bar{z} - w)} \quad (92)$$

The rationale for the i^n in (91) is twofold: first Theorem 11, second the fact that if the z_i 's are imaginary the function \mathcal{A}_σ and $i\mathcal{B}_\sigma$ obtained in Theorem 3 are real on $i\mathbb{R}$, hence \mathcal{E}_σ and \mathcal{F}_σ are real on $i\mathbb{R}$, hence $E_\sigma(it)$ and $F_\sigma(it)$ as defined by (91) are real for t real. The differential system with respect to x for E_σ and F_σ (as for their complete versions $\mathcal{E}_\sigma, \mathcal{F}_\sigma$) is:

$$\frac{d}{dx} E_\sigma(it) = t E_\sigma(it) + \mu_\sigma(x) F_\sigma(it) \quad (93a)$$

$$\frac{d}{dx} F_\sigma(it) = -t F_\sigma(it) + \mu_\sigma(x) E_\sigma(it) \quad (93b)$$

We introduce the coefficients $c_1, \dots, c_n, d_1, \dots, d_n$ which are the functions of x and of the imaginary points $z_1 = -i\kappa_1, \dots, z_n = -i\kappa_n$ such that, according to Theorem 11, the following holds:

$$E_\sigma(it) = e^{xt} + \sum_{1 \leq j \leq n} c_j \frac{2 \operatorname{sh}((t - \kappa_j)x)}{t - \kappa_j} \quad (94a)$$

$$F_\sigma(it) = (-1)^n e^{-xt} + \sum_{1 \leq j \leq n} d_j \frac{2 \operatorname{sh}((t - \kappa_j)x)}{t - \kappa_j} \quad (94b)$$

The identity following from $\mathcal{F}_\sigma = \mathcal{E}_\sigma^*$ is (we use that E_σ and F_σ are real valued on $i\mathbb{R}$):

$$\prod_j (t - \kappa_j) F_\sigma(it) = (-1)^n \prod_j (t + \kappa_j) E_\sigma(-it) \quad (95)$$

If one is interested in explicit formulas for the c_j 's and d_j 's, the initial recipe is to put $t = -\kappa_1, \dots, t = -\kappa_n$ in (94a) (resp. (94b)) and to use the trivial zeros $E_\sigma(-i\kappa_j) = 0$ (resp. $F_\sigma(-i\kappa_j) = 0$). Cramer's formulas thus lead to determinantal representations for the c_j 's and d_j 's (which are seen to be real valued).

Remark 6. We pause here to explain how to remove the restrictions $\kappa_i + \kappa_j \neq 0$. These constraints go back to Theorem 3. They were necessary to avoid vanishing of the denominators U_n and V_n , in the formulas for $\mathcal{A}_n, \mathcal{B}_n$. But (94a) and (94b) define E_σ and F_σ , and the validity of

$$K^\sigma(z, w) = \frac{\overline{E_\sigma(z)} E_\sigma(w) - \overline{F_\sigma(z)} F_\sigma(w)}{i(\bar{z} - w)} \quad (96)$$

follows by continuity (for real κ_i 's), as there is no singularity arising in the formulas for the coefficients $c_1, \dots, c_n, d_1, \dots, d_n$. The same remark applies to the mu-function μ_σ which will be expressed below in terms of these coefficients. Hence by continuity we again have a mu-function and a differential system (93a), (93b) even when $\kappa_i + \kappa_j = 0$ for some (i, j) . The conditions $\kappa_i + \kappa_j \neq 0$ were enforced only in order to facilitate the writing of explicit formulas of Wronskian type for the \mathcal{A} 's and \mathcal{B} 's.

There is a plethora of various algebraic and differential identities involving the c_j 's and d_j 's. We propose a basic selection, sufficient for our goal in this chapter. From (94a), the value of $(\frac{d}{dx} - t)E_\sigma(it)$ is

$$\sum_{1 \leq j \leq n} (c'_j - \kappa_j c_j) \frac{2 \operatorname{sh}((t - \kappa_j)x)}{t - \kappa_j} + \sum_{1 \leq j \leq n} c_j (2 \operatorname{ch}((t - \kappa_j)x) - 2 \operatorname{sh}((t - \kappa_j)x)) \quad (97)$$

Comparison with (93a) gives:

$$\mu_\sigma(x) = (-1)^n \sum_{1 \leq j \leq n} 2c_j e^{\kappa_j x} \quad (98)$$

$$\text{and } 1 \leq j \leq n \implies \frac{d}{dx}c_j - \kappa_j c_j = \mu_\sigma d_j \quad (99)$$

Similarly, from (94b), the value of $(\frac{d}{dx} + t)F_\sigma(it)$ is

$$\sum_{1 \leq j \leq n} (d'_j + \kappa_j d_j) \frac{2 \operatorname{sh}((t - \kappa_j)x)}{t - \kappa_j} + \sum_{1 \leq j \leq n} d_j (2 \operatorname{ch}((t - \kappa_j)x) + 2 \operatorname{sh}((t - \kappa_j)x)) \quad (100)$$

Thus:

$$\mu_\sigma(x) = \sum_{1 \leq j \leq n} 2d_j e^{-\kappa_j x} \quad (101)$$

$$\text{and } 1 \leq j \leq n \implies \frac{d}{dx}d_j + \kappa_j d_j = \mu_\sigma c_j \quad (102)$$

We take note of the asymptotics:

$$E_\sigma(it) =_{t \rightarrow +\infty} e^{xt} \left(1 - \frac{\alpha_\sigma(x)}{2t} + O(t^{-2}) \right) \quad \alpha_\sigma(x) = -2 \sum_{1 \leq j \leq n} c_j e^{-\kappa_j x} \quad (103a)$$

$$F_\sigma(it) =_{t \rightarrow -\infty} (-1)^n e^{-xt} \left(1 + \frac{\delta_\sigma(x)}{2t} + O(t^{-2}) \right) \quad \delta_\sigma(x) = -(-1)^n 2 \sum_{1 \leq j \leq n} d_j e^{\kappa_j x} \quad (103b)$$

Using (95) we obtain $\delta_\sigma(x) = \alpha_\sigma(x) + 4 \sum_{1 \leq j \leq n} \kappa_j$. Further,

$$\frac{d}{dx} \alpha_\sigma(x) = -2 \sum_{1 \leq j \leq n} (c'_j - \kappa_j c_j) e^{-\kappa_j x} = -2\mu_\sigma \sum_{1 \leq j \leq n} d_j e^{-\kappa_j x} = -\mu_\sigma^2 \quad (104)$$

Using either the differential equations or the identities already known provides the two further asymptotics:

$$F_\sigma(it) =_{t \rightarrow +\infty} e^{xt} \left(\frac{\mu_\sigma(x)}{2t} + O(t^{-2}) \right) \quad (105a)$$

$$E_\sigma(it) =_{t \rightarrow -\infty} (-1)^n e^{-xt} \left(\frac{-\mu_\sigma(x)}{2t} + O(t^{-2}) \right) \quad (105b)$$

We definitely switch to viewing functions depending on x as functions of the variable $a = e^{-x}$. For example we write $\mu_\sigma(a)$, rather than $\mu_\sigma(-\log(a))$. We have $a \frac{d}{da} = -\frac{d}{dx}$. We also fix once and for all an integer $n \geq 1$, and will study the spaces $PW_x(\nu, n)$ associated to a sequence of trivial zeros z_1, \dots, z_n , in arithmetic progression:

$$\kappa_1 = \frac{\nu+1}{2}, \kappa_2 = \frac{\nu+1}{2} + 1, \dots, \kappa_n = \frac{\nu+1}{2} + n - 1 \quad z_j = -i\kappa_j \quad (106)$$

The transition from n to $n+1$ is described by Theorem 10. Here n is fixed, and we shall study the relation between ν and $\nu+1$.

We will use the notations E_ν , $E_{\nu+1}$, F_ν , $F_{\nu+1}$, μ_ν , $\mu_{\nu+1}$, and K^ν , $K^{\nu+1}$ for the incomplete reproducing kernel. Neither the dependency on a nor on n is explicitly recalled in the notation. Also we shall write c_1^ν, \dots, c_n^ν and d_1^ν, \dots, d_n^ν , respectively $c_1^{\nu+1}, \dots, c_n^{\nu+1}$, and $d_1^{\nu+1}, \dots, d_n^{\nu+1}$, for the coefficients appearing in equations (94a) and (94b) for ν and $\nu+1$. These coefficients are functions of a (depending on n). We rewrite (94a) and (94b) as

$$E_\nu(it) = e^{xt} + \int_{-x}^x e^{-ty} e^{\frac{\nu+1}{2}y} \sum_{1 \leq j \leq n} c_j^\nu e^{(j-1)y} dy \quad (107a)$$

$$F_\nu(it) = (-1)^n e^{-xt} + \int_{-x}^x e^{-ty} e^{\frac{\nu+1}{2}y} \sum_{1 \leq j \leq n} d_j^\nu e^{(j-1)y} dy \quad (107b)$$

According to (107a):

$$a^{\frac{1}{2}} E_\nu(i(t + \frac{1}{2})) = e^{xt} + \int_{-x}^x e^{-ty} e^{\frac{\nu+2}{2}y} \sum_{0 \leq j \leq n-1} a^{\frac{1}{2}} c_{j+1}^\nu e^{(j-1)y} dy \quad (108)$$

So the function $w \mapsto a^{\frac{1}{2}} E_\nu(w + i\frac{1}{2}) - E_{\nu+1}(w)$ is a finite linear combination of the $n+1$ initial Paley-Wiener evaluators $Z(-i\frac{\nu}{2}, w)$, $Z(-i\frac{\nu+2}{2}, w)$, \dots , $Z(-i\frac{\nu+2}{2} - i(n-1), w)$. Moreover it has trivial zeros at the trivial zeros of $E_{\nu+1}$. By Theorem 2 this identifies $a^{\frac{1}{2}} E_\nu(w + i\frac{1}{2}) - E_{\nu+1}(w)$ with a constant multiple of $K^{\nu+1}(-i\frac{\nu}{2}, w)$, the factor being precisely $a^{\frac{1}{2}} c_1^\nu$. We prove in this manner the first of the following identities:

Proposition 12. *There holds*

$$a^{\frac{1}{2}} E_\nu(w + i\frac{1}{2}) = E_{\nu+1}(w) + a^{\frac{1}{2}} c_1^\nu K^{\nu+1}(-i\frac{\nu}{2}, w) \quad (109a)$$

$$a^{-\frac{1}{2}} F_\nu(w + i\frac{1}{2}) = F_{\nu+1}(w) + a^{-\frac{1}{2}} d_1^\nu K^{\nu+1}(-i\frac{\nu}{2}, w) \quad (109b)$$

$$a^{-\frac{1}{2}} E_{\nu+1}(w - i\frac{1}{2}) = E_\nu(w) + a^{-\frac{1}{2}} c_n^{\nu+1} K^\nu(-i\frac{\nu+1}{2} - in, w) \quad (109c)$$

$$a^{\frac{1}{2}} F_{\nu+1}(w - i\frac{1}{2}) = F_\nu(w) + a^{\frac{1}{2}} d_n^{\nu+1} K^\nu(-i\frac{\nu+1}{2} - in, w) \quad (109d)$$

Proof. The additional three identities are proved by the same method as the first. We will not make direct use of the ensuing relation between the two kinds of evaluators. \square

Let us recall that:

$$K^{\nu+1}(-i\frac{\nu}{2}, it) = \frac{E_{\nu+1}(-i\frac{\nu}{2})E_{\nu+1}(it) - F_{\nu+1}(-i\frac{\nu}{2})F_{\nu+1}(it)}{t - \frac{\nu}{2}} \quad (110a)$$

$$K^\nu(-i(\frac{\nu+1}{2} + n), it) = \frac{E_\nu(-i(\frac{\nu+1}{2} + n))E_\nu(it) - F_\nu(-i(\frac{\nu+1}{2} + n))F_\nu(it)}{t - \frac{\nu+1}{2} - n} \quad (110b)$$

In order to shorten the formulas we adopt the notations:

$$e_\nu = E_\nu(-i\frac{\nu+1}{2} - in) \quad g_{\nu+1} = E_{\nu+1}(-i\frac{\nu}{2}) \quad (111a)$$

$$f_\nu = F_\nu(-i\frac{\nu+1}{2} - in) \quad h_{\nu+1} = F_{\nu+1}(-i\frac{\nu}{2}) \quad (111b)$$

Combining (109a) and (110a) gives (with the notation $\text{sh}_x(y) = \frac{\text{sh}(xy)}{y}$)

$$\begin{aligned} (t - \frac{\nu}{2}) \left(\sum_{0 \leq j \leq n-1} a^{\frac{1}{2}} c_{j+1}^\nu 2 \text{sh}_x(t - \frac{\nu}{2} - j) - \sum_{1 \leq j \leq n} c_j^{\nu+1} 2 \text{sh}_x(t - \frac{\nu}{2} - j) \right) \\ = a^{\frac{1}{2}} c_1^\nu (g_{\nu+1} E_{\nu+1}(it) - h_{\nu+1} F_{\nu+1}(it)) \end{aligned} \quad (112)$$

Using $(t - \frac{\nu}{2}) \text{sh}_x(t - \frac{\nu}{2} - j) = \text{sh}(x(t - \frac{\nu}{2} - j)) + j \text{sh}_x(t - \frac{\nu}{2} - j)$ we can rearrange and obtain identities by termwise identifications. We record only the identity corresponding to the term with the highest value of j ($j = n$):

$$n c_n^{\nu+1} = a^{\frac{1}{2}} c_1^\nu (h_{\nu+1} d_n^{\nu+1} - g_{\nu+1} c_n^{\nu+1}) \quad (113)$$

Combining similarly (109b) with (110a) we obtain (among other identities!)

$$n d_n^{\nu+1} = a^{-\frac{1}{2}} d_1^\nu (h_{\nu+1} d_n^{\nu+1} - g_{\nu+1} c_n^{\nu+1}) \quad (114)$$

Dealing in the same manner with (109c) and (109d) and (110b) gives two further identities of a symmetric type. Summing up, the four obtained relations are:

Proposition 13. *There holds:*

$$n c_n^{\nu+1} = a^{\frac{1}{2}} c_1^\nu (h_{\nu+1} d_n^{\nu+1} - g_{\nu+1} c_n^{\nu+1}) \quad (115a)$$

$$n d_n^{\nu+1} = a^{-\frac{1}{2}} d_1^\nu (h_{\nu+1} d_n^{\nu+1} - g_{\nu+1} c_n^{\nu+1}) \quad (115b)$$

$$-n c_1^\nu = a^{-\frac{1}{2}} c_n^{\nu+1} (f_\nu d_1^\nu - e_\nu c_1^\nu) \quad (115c)$$

$$-n d_1^\nu = a^{\frac{1}{2}} d_n^{\nu+1} (f_\nu d_1^\nu - e_\nu c_1^\nu) \quad (115d)$$

We will need the following:

Proposition 14. *None of the eight quantities e_ν , f_ν , $g_{\nu+1}$, $h_{\nu+1}$, c_1^ν , d_1^ν , $c_{n+1}^{\nu+1}$, and $d_{n+1}^{\nu+1}$ can vanish.*

Proof. The proof introduces tacitly a link with techniques of orthogonal polynomial theory, which leads to some results we will expose elsewhere. Let us write explicitly the system of linear equations for the c_ν^j 's:

$$\forall i = 1 \dots n \quad \sum_{1 \leq j \leq n} c_j^\nu \int_{-x}^x e^{(\kappa_i + \kappa_j)y} dy = -e^{-x\kappa_i} \quad (116)$$

where we recall that $\kappa_j = \frac{\nu+1}{2} + j - 1$. Let G be the matrix of this system, we have:

$$\det(G)c_1^\nu = \begin{vmatrix} -e^{-x\kappa_1} & \int_{-x}^x e^{(\kappa_1+\kappa_2)y} dy & \dots & \int_{-x}^x e^{(\kappa_1+\kappa_n)y} dy \\ -e^{-x\kappa_2} & \int_{-x}^x e^{(\kappa_2+\kappa_2)y} dy & \dots & \int_{-x}^x e^{(\kappa_2+\kappa_n)y} dy \\ \dots & \dots & \dots & \dots \\ -e^{-x\kappa_n} & \int_{-x}^x e^{(\kappa_n+\kappa_2)y} dy & \dots & \int_{-x}^x e^{(\kappa_n+\kappa_n)y} dy \end{vmatrix} \quad (117)$$

We now exploit the relation $\kappa_{j+1} = \kappa_j + 1$ to transform by row manipulations the determinant on the right into:

$$\begin{vmatrix} -e^{-x\kappa_1} & \int_{-x}^x e^{(\kappa_1+\kappa_2)y} dy & \dots & \int_{-x}^x e^{(\kappa_1+\kappa_n)y} dy \\ 0 & \int_{-x}^x e^{(\kappa_2+\kappa_2)y} (1 - e^{-x} e^{-y}) dy & \dots & \int_{-x}^x e^{(\kappa_2+\kappa_n)y} (1 - e^{-x} e^{-y}) dy \\ 0 & \dots & \dots & \dots \\ 0 & \int_{-x}^x e^{(\kappa_n+\kappa_2)y} (1 - e^{-x} e^{-y}) dy & \dots & \int_{-x}^x e^{(\kappa_n+\kappa_n)y} (1 - e^{-x} e^{-y}) dy \end{vmatrix} \\ = -e^{-x\kappa_1} \begin{vmatrix} \int_{-x}^x e^{(\kappa_2+\kappa_2)y} (e^y - e^{-x}) e^{-y} dy & \dots & \int_{-x}^x e^{(\kappa_2+\kappa_n)y} (e^y - e^{-x}) e^{-y} dy \\ \dots & \dots & \dots \\ \int_{-x}^x e^{(\kappa_n+\kappa_2)y} (e^y - e^{-x}) e^{-y} dy & \dots & \int_{-x}^x e^{(\kappa_n+\kappa_n)y} (e^y - e^{-x}) e^{-y} dy \end{vmatrix} \quad (118)$$

This new determinant is a Gramian for a positive measure, it is strictly positive. So $c_1^\nu < 0$. It can be proven in the exact same manner $d_1^\nu > 0$, $(-1)^n c_n^{\nu+1} > 0$, $(-1)^n d_n^{\nu+1} < 0$, $(-1)^n e_\nu > 0$, $(-1)^n f_\nu > 0$, $g_{\nu+1} > 0$, and $h_{\nu+1} > 0$. \square

Using the asymptotics for $t \rightarrow -\infty$ (103b) and (105b) in (109a) (and (110a)) and also in (109c) (and (110b)), and also the asymptotics (103a), (105a) ($t \rightarrow +\infty$) in (109b) and (109d) give the following proposition.

Proposition 15. *There holds*

$$a\mu_\nu - \mu_{\nu+1} = 2a^{\frac{1}{2}} c_1^\nu h_{\nu+1} = -2a^{\frac{1}{2}} c_n^{\nu+1} f_\nu \quad (119a)$$

$$a\mu_{\nu+1} - \mu_\nu = -2a^{\frac{1}{2}} d_1^\nu g_{\nu+1} = 2a^{\frac{1}{2}} d_n^{\nu+1} e_\nu \quad (119b)$$

Proposition 16. *There holds:*

$$c_1^\nu h_{\nu+1} = -c_n^{\nu+1} f_\nu \quad (120a)$$

$$d_1^\nu g_{\nu+1} = -d_n^{\nu+1} e_\nu \quad (120b)$$

$$a c_1^\nu g_{\nu+1} = -c_n^{\nu+1} e_\nu \quad (120c)$$

$$d_1^\nu h_{\nu+1} = -a d_n^{\nu+1} f_\nu \quad (120d)$$

Proof. The first two follow from the previous proposition. The last two follow from the first and the relation $\frac{ac_1^\nu}{d_1^\nu} = \frac{c_n^{\nu+1}}{d_n^{\nu+1}}$ (from (115a), (115b)). Alternatively they can be deduced from a look at the asymptotics in (109a), (109c) for $w = it$, $t \rightarrow +\infty$ and in (109b) and (109d) for $t \rightarrow -\infty$. \square

Definition 1. We define the quantities $X = X(\nu, n, a)$ and $Y(\nu, n, a)$ by the following expressions:

$$X := \frac{g_{\nu+1}}{h_{\nu+1}} = \frac{1}{a} \frac{e_\nu}{f_\nu} \quad (121a)$$

$$Y := -\frac{1}{a} \frac{c_n^{\nu+1}}{d_n^{\nu+1}} = -\frac{c_1^\nu}{d_1^\nu} \quad (121b)$$

From Proposition 15 we have $\mu_\nu = \frac{2a^{\frac{1}{2}}}{1-a^2} (-ah_{\nu+1}c_1^\nu + g_{\nu+1}d_1^\nu)$ and using the first two relations in Proposition 13 gives

$$\mu_\nu = \frac{2na}{1-a^2} \frac{-h_{\nu+1}c_n^{\nu+1} + g_{\nu+1}d_n^{\nu+1}}{h_{\nu+1}d_n^{\nu+1} - g_{\nu+1}c_n^{\nu+1}} = \frac{2na}{1-a^2} \frac{X + aY}{1 + aXY} \quad (122)$$

Similarly from (119a) and (119b) $\mu_{\nu+1} = \frac{2a^{\frac{1}{2}}}{1-a^2} (-h_{\nu+1}c_1^\nu + ag_{\nu+1}d_1^\nu)$ which gives using Proposition 13 and Definition 1

$$\mu_{\nu+1} = \frac{2na}{1-a^2} \frac{aX + Y}{1 + aXY} \quad (123)$$

Theorem 17. The quantities X and Y from Definition 1 are related to the mu-functions μ_ν and $\mu_{\nu+1}$ by the equations:

$$\mu_\nu = \frac{2na}{1-a^2} \frac{X + aY}{1 + aXY} \quad \mu_{\nu+1} = \frac{2na}{1-a^2} \frac{aX + Y}{1 + aXY} \quad (124)$$

They obey the following non-linear differential system:

$$a \frac{d}{da} X = \nu X - (1 - X^2) \frac{2na}{1-a^2} \frac{aX + Y}{1 + aXY} \quad (125a)$$

$$a \frac{d}{da} Y = -(\nu + 1)Y + (1 - Y^2) \frac{2na}{1-a^2} \frac{X + aY}{1 + aXY} \quad (125b)$$

Proof. From $X = \frac{g_{\nu+1}}{h_{\nu+1}}$ and (93a), (93b):

$$-a \frac{d}{da} g_{\nu+1} = -\frac{\nu}{2} g_{\nu+1} + \mu_{\nu+1} h_{\nu+1} \quad (126a)$$

$$-a \frac{d}{da} h_{\nu+1} = \frac{\nu}{2} h_{\nu+1} + \mu_{\nu+1} g_{\nu+1} \quad (126b)$$

Hence: $a \frac{d}{da} X = \nu X - \mu_{\nu+1}(1 - X^2)$. And from $Y = -\frac{c_1^\nu}{d_1^\nu}$ and (99), (102) we have:

$$-a \frac{d}{da} c_1^\nu = \frac{\nu + 1}{2} c_1^\nu + \mu_\nu d_1^\nu \quad (127a)$$

$$-a \frac{d}{da} d_1^\nu = -\frac{\nu + 1}{2} d_1^\nu + \mu_\nu c_1^\nu \quad (127b)$$

and this gives $a \frac{d}{da} Y = -(\nu + 1)Y + \mu_\nu(1 - Y^2)$. \square

Theorem 18. *Let (X, Y) be two functions of a variable a . If they obey the differential system $(VI_{\nu, n})$:*

$$a \frac{d}{da} X = \nu X - (1 - X^2) \frac{2na}{1 - a^2} \frac{aX + Y}{1 + aXY} \quad (128a)$$

$$a \frac{d}{da} Y = -(\nu + 1)Y + (1 - Y^2) \frac{2na}{1 - a^2} \frac{X + aY}{1 + aXY} \quad (128b)$$

then the quantity $q = a \frac{aX+Y}{X+aY}$ satisfies as function of $b = a^2$ the P_{VI} differential equation:

$$\begin{aligned} \frac{d^2 q}{db^2} = & \frac{1}{2} \left\{ \frac{1}{q} + \frac{1}{q-1} + \frac{1}{q-b} \right\} \left(\frac{dq}{db} \right)^2 - \left\{ \frac{1}{b} + \frac{1}{b-1} + \frac{1}{q-b} \right\} \frac{dq}{db} \\ & + \frac{q(q-1)(q-b)}{b^2(b-1)^2} \left\{ \alpha + \frac{\beta b}{q^2} + \frac{\gamma(b-1)}{(q-1)^2} + \frac{\delta b(b-1)}{(q-b)^2} \right\} \end{aligned} \quad (129)$$

with parameters $(\alpha, \beta, \gamma, \delta) = (\frac{(\nu+n)^2}{2}, \frac{-(\nu+n+1)^2}{2}, \frac{n^2}{2}, \frac{1-n^2}{2})$.

Proof. The computation being lengthy we only give some brief indications. It is useful to introduce the variable

$$T := \frac{1}{1 + aXY} \quad (130)$$

It verifies the differential equation:

$$a \frac{d}{da} T = \frac{2a^2 n}{1 - a^2} T^2 (Y^2 - X^2) \quad (131)$$

Let $b = a^2$. One has

$$\frac{d}{db} T = T(T-1) \frac{n(q^2 - b)}{b(q-b)(q-1)} \quad (132)$$

On the other hand one establishes:

$$\frac{dq}{db} = \frac{2n + \nu + (\nu + 1)b}{b(b-1)} q - \frac{(\nu + n)q^2 + (\nu + 1 + n)b}{b(b-1)} + \frac{2nqT}{b} \quad (133)$$

This equation gives an expression of T in terms of $\frac{dq}{db}$, q , and b . Substituting this value of T in (132) gives a second order differential equation for q . With some tenacity one finally realizes that it is nothing else than P_{VI} with parameters $(\alpha, \beta, \gamma, \delta) = (\frac{(\nu+n)^2}{2}, \frac{-(\nu+n+1)^2}{2}, \frac{n^2}{2}, \frac{1-n^2}{2})$. \square

Remark 7. One also establishes that as functions of $b = a^2$, the expressions $\frac{-aY}{X}$ and $a^2 X^2$ are P_{VI} -transcendents.

We mention finally the following:

Theorem 19. Let (X, Y) be a solution of the non-linear system $(VI_{\nu,n})$. Let Z be defined by the following relation ($n \neq 1$):

$$\frac{aY + Z}{1 + aYZ} = \left(\frac{1}{a} - a\right) \frac{\nu + 1}{n - 1} \frac{Y}{1 - Y^2} - \frac{n}{(n - 1)} \frac{aY + X}{1 + aXY} \quad (134)$$

Then (Y, Z) is a solution of system $(VI_{\nu+1,n-1})$. Let W be such that

$$\frac{W + aX}{1 + aWX} = \left(\frac{1}{a} - a\right) \frac{\nu}{n + 1} \frac{X}{1 - X^2} - \frac{n}{(n + 1)} \frac{aX + Y}{1 + aXY} \quad (135)$$

Then (W, X) is a solution of system $(VI_{\nu-1,n+1})$.

Proof. One needs to eliminate X (resp. Y) from the result of computing $\frac{dZ}{da}$ (resp. $\frac{dW}{da}$). This is a long computation. \square

7 Conclusion

We assemble some of our main results in the following summary:

Theorem 20. Let $x > 0$, $a = e^{-x}$, and let PW_x be the Paley-Wiener space of entire functions which are square integrable on the real line and of finite exponential type at most $x > 0$. Let

$$\sigma = (z_1 = -i\kappa_1, \dots, z_n = -i\kappa_n) \quad (136)$$

be a finite sequence of n distinct purely imaginary numbers and

$$PW_x(\sigma) = \left\{ \frac{f(z)}{\prod_{1 \leq j \leq n} (z - z_j)} \mid f \in PW_x, f(z_1) = \dots = f(z_n) = 0 \right\} \quad (137)$$

The modified space $PW_x(\sigma)$ is a Hilbert space through its identification with the closed subspace of functions in PW_x vanishing on σ . There is a unique entire function \mathcal{E}_σ verifying the conditions:

1. $\mathcal{E}_\sigma(it)$ is real for t real,
2. $\mathcal{E}_\sigma(0) > 0$,
3. $\lim_{t \rightarrow +\infty} \frac{\mathcal{E}_\sigma(-it)}{\mathcal{E}_\sigma(it)} = 0$,

and in terms of which the scalar products of evaluators in $PW_x(\sigma)$ are:

$$\mathcal{K}_\sigma(z, w) = \frac{\overline{\mathcal{E}_\sigma(z)} \mathcal{E}_\sigma(w) - \mathcal{E}_\sigma(\bar{z}) \overline{\mathcal{E}_\sigma(\bar{w})}}{i(\bar{z} - w)} \quad (138)$$

Let $\mathcal{F}_\sigma(w) = \overline{\mathcal{E}_\sigma(\overline{w})}$ ($= \mathcal{E}_\sigma(-w)$). There holds:

$$-a \frac{d}{da} \mathcal{E}_\sigma(w) = -iw \mathcal{E}_\sigma(w) + \mu_\sigma(a) \mathcal{F}_\sigma(w) \quad (139)$$

The function $\mu_\sigma(a)$ admits (among others) the two representations:

$$\mu_\sigma(a) = a \frac{d}{da} \log \frac{W(\text{ch}(\kappa_1 x), \dots, \text{ch}(\kappa_n x))}{W(\text{sh}(\kappa_1 x), \dots, \text{sh}(\kappa_n x))} \quad (140)$$

where $W(f_1, \dots, f_n)$ is the Wronskian determinant (with respect to $x = \log \frac{1}{a}$) and

$$\mu_\sigma(a) = \frac{2(-1)^n}{g_n(a)} \begin{vmatrix} \int_{-x}^x e^{(\kappa_1 + \kappa_1)y} dy & \int_{-x}^x e^{(\kappa_1 + \kappa_2)y} dy & \dots & \int_{-x}^x e^{(\kappa_1 + \kappa_n)y} dy & e^{-x\kappa_1} \\ \int_{-x}^x e^{(\kappa_2 + \kappa_1)y} dy & \int_{-x}^x e^{(\kappa_2 + \kappa_2)y} dy & \dots & \int_{-x}^x e^{(\kappa_2 + \kappa_n)y} dy & e^{-x\kappa_2} \\ \dots & \dots & \dots & \dots & \dots \\ \int_{-x}^x e^{(\kappa_n + \kappa_1)y} dy & \int_{-x}^x e^{(\kappa_n + \kappa_2)y} dy & \dots & \int_{-x}^x e^{(\kappa_n + \kappa_n)y} dy & e^{-x\kappa_n} \\ e^{x\kappa_1} & \dots & \dots & e^{x\kappa_n} & 0 \end{vmatrix} \quad (141)$$

where $g_n(a) = \det(\int_{-x}^x e^{(\kappa_i + \kappa_j)y} dy)_{1 \leq i, j \leq n}$ is the principal $n \times n$ minor of the $(n+1) \times (n+1)$ determinant. There also holds

$$\mu_\sigma(a)^2 = -a \frac{d}{da} a \frac{d}{da} \log g_n(a) \quad (142)$$

In the specific case where σ is an arithmetic progression:

$$\kappa_1 = \frac{\nu+1}{2}, \kappa_2 = \frac{\nu+1}{2} + 1, \dots, \kappa_n = \frac{\nu+1}{2} + n - 1 \quad (143)$$

the function $\mu_{\nu,n}(a)$ can be expressed as a quotient of two multiple integrals:

$$\mu_{\nu,n}(a) = 2n \frac{\int \dots \int_{[a, \frac{1}{a}]^{n-1}} \prod_{i < j} (t_j - t_i)^2 \prod_i t_i^\nu (t_i - a) (\frac{1}{a} - t_i) dt_1 \dots dt_{n-1}}{\int \dots \int_{[a, \frac{1}{a}]^n} \prod_{i < j} (t_j - t_i)^2 \prod_i t_i^\nu dt_1 \dots dt_n} \quad (144)$$

and the expression

$$q_{\nu,n} = \frac{a\mu_{\nu+1,n}}{\mu_{\nu,n}}, \quad (145)$$

as a function of a^2 , verifies the Painlevé VI equation with parameters

$$(\alpha, \beta, \gamma, \delta) = \left(\frac{(\nu+n)^2}{2}, \frac{-(\nu+n+1)^2}{2}, \frac{n^2}{2}, \frac{1-n^2}{2} \right) \quad (146)$$

Proof. We have established (138) with $\mathcal{E}_\sigma(it) = \prod_{1 \leq i \leq n} \frac{1}{t + \kappa_i} E_\sigma(it)$ and $E_\sigma(it)$ given by equation (94a). In particular E_σ hence \mathcal{E}_σ is real on the imaginary axis. One has $E_\sigma(it) \sim_{t \rightarrow +\infty} e^{xt}$, hence $\mathcal{E}_\sigma(it)$, which by (138) cannot vanish for $t > 0$, is positive for $t > 0$. It is easy to prove that given arbitrary points w_1, \dots, w_m and non-negative integers n_1, \dots, n_m there is in PW_x a function vanishing exactly to the order n_j at w_j for all j . So in $PW_x(\sigma)$ the

evaluator at $z = 0$ is non zero, which proves $\mathcal{E}_\sigma(0) \neq 0$, hence $\mathcal{E}_\sigma(0) > 0$. As \mathcal{E}_σ is real on the imaginary line one has $\mathcal{F}_\sigma(w) = \mathcal{E}_\sigma(-w)$. And from the representations (94a) and (94b) we know $\frac{\mathcal{F}_\sigma(it)}{\mathcal{E}_\sigma(it)} \rightarrow_{t \rightarrow +\infty} 0$. Let \mathcal{E} be another function computing the reproducing kernel and with the properties (1), (2) and (3). Let $\mathcal{A} = \frac{1}{2}(\mathcal{E} + \mathcal{E}^*)$, $\mathcal{B} = \frac{i}{2}(\mathcal{E} - \mathcal{E}^*)$, which are respectively even and odd, real on the real line. The evaluator at the origin (as said above, necessarily non zero) is $\mathcal{K}^\sigma(0, w) = 2\mathcal{A}(0)\frac{1}{w}\mathcal{B}(w)$, hence the function \mathcal{B} is known up to a positive real multiple, then the function \mathcal{A} is known up to (the inverse of) this multiple. Condition (3) can be written $\frac{-i\mathcal{B}(it)}{\mathcal{A}(it)} \rightarrow_{t \rightarrow +\infty} 1$, and this finally identifies $\mathcal{A} = \mathcal{A}_\sigma$ and $\mathcal{B} = \mathcal{B}_\sigma$.

Formula (140) (in which $\kappa_i + \kappa_j \neq 0$ is assumed) follows from Theorem 9, and formula (142) from Theorem 9 and Proposition 4. According to (98) one has $\mu_\sigma(a) = 2(-1)^n \sum_{1 \leq j \leq n} c_j^\sigma e^{\kappa_j x}$, the coefficients c_j^σ solving:

$$\forall i = 1 \dots n \quad \sum_{1 \leq j \leq n} c_j^\sigma \int_{-x}^x e^{(\kappa_i + \kappa_j)y} dy = -e^{-x\kappa_i} \quad (147)$$

This gives the representation (141).

In the case of the arithmetic progression of reason one, row manipulations replace the $(n+1) \times (n+1)$ determinant by

$$-e^{-x\kappa_n} \begin{vmatrix} \int_{-x}^x e^{(\kappa_1 + \kappa_1)y} (1 - \frac{1}{a}e^y) dy & \dots & \int_{-x}^x e^{(\kappa_1 + \kappa_n)y} (1 - \frac{1}{a}e^y) dy \\ \int_{-x}^x e^{(\kappa_2 + \kappa_1)y} (1 - \frac{1}{a}e^y) dy & \dots & \int_{-x}^x e^{(\kappa_2 + \kappa_n)y} (1 - \frac{1}{a}e^y) dy \\ \dots & \dots & \dots \\ \int_{-x}^x e^{(\kappa_{n-1} + \kappa_1)y} (1 - \frac{1}{a}e^y) dy & \dots & \int_{-x}^x e^{(\kappa_{n-1} + \kappa_n)y} (1 - \frac{1}{a}e^y) dy \\ e^{x\kappa_1} & \dots & e^{x\kappa_n} \end{vmatrix} \quad (148)$$

Column manipulations lead to:

$$\begin{aligned} & - \begin{vmatrix} \int_{-x}^x e^{(\kappa_1 + \kappa_1)y} (1 - ae^y) (1 - \frac{1}{a}e^y) dy & \dots & \int_{-x}^x e^{(\kappa_1 + \kappa_{n-1})y} (1 - ae^y) (1 - \frac{1}{a}e^y) dy \\ \int_{-x}^x e^{(\kappa_2 + \kappa_1)y} (1 - ae^y) (1 - \frac{1}{a}e^y) dy & \dots & \int_{-x}^x e^{(\kappa_2 + \kappa_{n-1})y} (1 - ae^y) (1 - \frac{1}{a}e^y) dy \\ \dots & \dots & \dots \\ \int_{-x}^x e^{(\kappa_{n-1} + \kappa_1)y} (1 - ae^y) (1 - \frac{1}{a}e^y) dy & \dots & \int_{-x}^x e^{(\kappa_{n-1} + \kappa_{n-1})y} (1 - ae^y) (1 - \frac{1}{a}e^y) dy \end{vmatrix} \\ & = (-1)^n \det_{1 \leq i, j \leq n-1} \left(\int_a^{\frac{1}{a}} t^{\nu+i+j-2} \left(\frac{1}{a} - t \right) (t - a) dt \right) \end{aligned} \quad (149)$$

So we have the representation of $\mu_{\nu, n}$ as a quotient of two gramians (for $n = 1$ the determinant at the numerator is taken to be 1):

$$\mu_{\nu, n}(a) = 2 \frac{\det_{1 \leq i, j \leq n-1} \left(\int_a^{\frac{1}{a}} t^{\nu+i+j-2} \left(\frac{1}{a} - t \right) (t - a) dt \right)}{\det_{1 \leq i, j \leq n} \left(\int_a^{\frac{1}{a}} t^{\nu+i+j-2} dt \right)} \quad (150)$$

This gives formula (144).

Finally the Painlevé VI assertion follows from Theorems 17 and 18. \square

Remark 8. The change of variables $t_j = a + (\frac{1}{a} - a)u$ transforms the multiple integrals at the numerator and denominator of expression (144) into, respectively:

$$(\frac{1}{a} - a)^{n^2-1} a^{(n-1)\nu} \int \cdots \int_{[0,1]^{n-1}} \prod_{i < j} (u_j - u_i)^2 \prod_i (1 + \alpha u_i)^\nu u_i (1 - u_i) du_1 \dots du_{n-1} \quad (151)$$

$$\text{and } (\frac{1}{a} - a)^{n^2} a^{n\nu} \int \cdots \int_{[0,1]^n} \prod_{i < j} (u_j - u_i)^2 \prod_i (1 + \alpha u_i)^\nu du_1 \dots du_n \quad (152)$$

with $\alpha = \frac{1}{a^2} - 1$. Similar multiple integrals arise in the work of Forrester and Witte [8] relating the “Jacobi unitary ensemble” of random matrices to the Okamoto hamiltonian formulation of the Painlevé VI equation [13]. It seems however (cf. the introduction of [9]) that in this context of multiple integrals one had so far not yet encountered directly Painlevé VI transcendents *per se*, but rather solutions to Okamoto’s “ σ -equation”.

Remark 9. We see from (150) or (141) that $\mu_{\nu,n}(a)$ is the product of $a^{\nu+1}$ with a rational function of a^2 and $a^{2\nu}$. The quantity $q = a^{\frac{\mu_{\nu+1,n}}{\mu_{\nu,n}}}$ is a rational function (with rational coefficients) of b and b^ν ($b = a^2$). In particular, for $\nu \in \mathbb{Z}$, q is a rational function of b .

Remark 10. We have left aside most of the developments which have their origins in [3], [4] and which put the objects studied here in another context (which leads in particular to various further representations for the μ -functions), indeed a context which presided over their introduction. We have also left aside a number of other developments related to techniques of orthogonal polynomial theory, multiple integrals and non-linear relations. We hope to address these topics in further publications.

Acknowledgments. All results included in this manuscript were obtained between January and late April 2008, while I was spending a seven months visit at the I.H.E.S. Numerous conversations on various (mostly other) topics with permanent members, visitors and staff made this a particularly stimulating and memorable residence. I also benefited during this period from electronic exchanges with Philippe Biane. I had found non-linear equations in the framework I had developed to establish the Fourier transform as a scattering [3, 4] and Philippe Biane’s question about what this would give when the Gamma function was replaced by a finite rational expression gave a welcome impetus to my efforts of that time. I discovered in the process of answering this question the non-linear equations (125a), (125b), from which the previous ones derived by a process of confluence, and I related them to Painlevé VI. I thus thank him for many interesting interactions (for example, in relation with the Wronskians built with the trigonometric functions, which he had also encountered in his own enterprises), and for having then kept me informed of some of his works [1]. I also thank the organizers Joaquim Burna, Håkan Hedenmalm, Kristian Seip, and Mikhail Sodin of the mini-symposium on Hilbert spaces of entire functions (EMS congress, Amsterdam 2008) for having

given me the opportunity to present there aspects of this material. Finally, I would like to express special thanks to Michel Balazard for having made possible my participation to the Zeta Functions congresses I, II, and III in Moscow, in 2006, 2008, and 2010, which gave me also opportunities to present related material.

References

- [1] Ph. Biane, *Orthogonal polynomials on the unit circle, q -Gamma weights, and discrete Painlevé equations*, arXiv:0901.0947, July 2010 (v1 January 2009), 26 pages.
- [2] L. de Branges, *Hilbert spaces of entire functions*, Prentice Hall Inc., Englewood Cliffs, 1968.
- [3] J.-F. Burnol, *Des équations de Dirac et de Schrödinger pour la transformation de Fourier*, C. R. Acad. Sci. Paris Ser. I, 336 (2003) 919-924.
- [4] J.-F. Burnol, *Scattering, determinants, hyperfunctions in relation to $\frac{\Gamma(1-s)}{\Gamma(s)}$* , 63 pages, feb 2006. arXiv:math.NT/0602425.
- [5] J.-F. Burnol, *Hilbert spaces of entire functions with trivial zeros*, 10 pages, July 2010. arXiv:
- [6] M. M. Crum, *Associated Sturm-Liouville systems*, Quart. J. Math. Oxford (2), 6 (1955) 121-127.
- [7] G. Darboux, *Sur une proposition relative aux équations linéaires*, Comptes Rendus Acad. Sci. 94 (1882) 1456-1459.
- [8] P.J. Forrester and N.S. Witte, *Application of the τ -function theory of Painlevé equations to random matrices: P_{VI} , the JUE, CyUE, cJUE and scaled limits*, Nagoya Math. J. Vol. 174 (2004) 29-114.
- [9] P.J. Forrester and N.S. Witte, *Random matrix theory and the sixth Painlevé equation*, J. Phys. A: Math. Gen. 39 (2006) 12211-12233.
- [10] M. G. Krein, *On the determination of the potential of a particle from its S -function*, Dokl. Akad. Nauk SSSR 105, no 3. (1955) 433-436.
- [11] M. G. Krein, *Continual analogues of propositions on polynomials orthogonal on the unit circle*, Dokl. Akad. Nauk SSSR 105, no 4. (1955) 637-640.
- [12] S. Okada, *Applications of Minor Summation Formulas to Rectangular-Shaped Representations of Classical Groups*, J. of Algebra, Vol. 205, no. 2 (1998) 337-367
- [13] K. Okamoto, *Studies on the Painlevé equations. I. Sixth Painlevé equation P_{VI}* , Ann. Mat. Pura Appl. (4), 146 (1987) 337-381.